The implicit equation of a canal surface

Marc Dohm, Severinas Zube

The implicit equation of a canal surface*

Marc Dohm, Severinas Zube

June 25, 2008

Abstract

A canal surface is an envelope of a one parameter family of spheres. In this paper we present an efficient algorithm for computing the implicit equation of a canal surface generated by a rational family of spheres. By using Laguerre and Lie geometries, we relate the equation of the canal surface to the equation of a dual variety of a certain curve in 5-dimensional projective space. We define the μ-basis for arbitrary dimension and give a simple algorithm for its computation. This is then applied to the dual variety, which allows us to deduce the implicit equations of the the dual variety, the canal surface and any offset to the canal surface.

Key words: canal surface, implicit equation, resultant, μ-basis, offset

1 Introduction

In surface design, the user often needs to perform rounding or filleting between two intersecting surfaces. Mathematically, the surface used in making the rounding is defined as the envelope of a family of spheres which are tangent to both surfaces. This envelope of spheres centered at $c(t) \in \mathbb{R}^3$ with radius $r(t)$, where $c(t)$ and $r(t)$ are rational functions, is called a canal surface with spine curve $\mathcal{E} = \{(c(t), r(t)) \in \mathbb{R}^4 | t \in \mathbb{R}\}$. If the radius $r(t)$ is constant the surface is called a pipe surface. Moreover, if additionally we reduce the dimension (take $c(t)$ in a plane and consider circles instead of spheres) we obtain the offset to the curve. Canal surfaces are very popular in Geometric Modelling, as they can be used as a blending surface between two surfaces. For example, any two circular cones with a common inscribed sphere can be blended by a part of a Dupin cyclide bounded by two circles as it was shown by [Pratt(1990), Pratt(1995)] (see Figure 1). Cyclides are envelopes of special quadratic families of spheres. For other examples of blending with canal surfaces we refer to [Kazakeviciute(2005)].

Here we study the implicit equation of a canal surface $\mathcal{C}$ and its implicit degree. The implicit equation of a canal surface can be obtained after elimination of the family variable t from the system of two equations $g_1(y,t) = g_2(y,t) = 0$ (here g_1, g_2 are quadratic in the variables $y = (y_1, y_2, y_3, y_4)$), i.e. by taking the resultant with respect to t. However, this resultant can have extraneous factors. In the paper we explain how these factors appear and how we can eliminate

*This is the accepted authors' manuscript of the text. The article is to be published in the Journal of Symbolic Computation, see http://dx.doi.org/10.1016/j.jsc.2008.06.001 for more information.

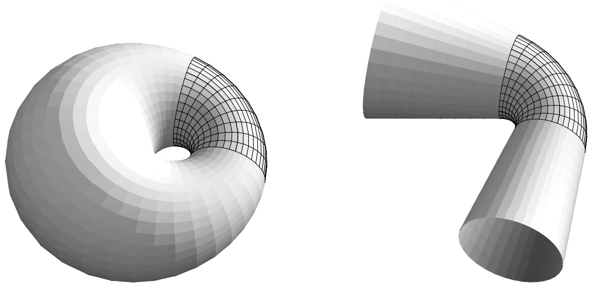

Figure 1: A Dupin cyclide used for blending circular cones.

them. By using Lie and Laguerre geometry, we see that the above system of equations is related to a system $h_1(\hat{y}, t) = h_2(\hat{y}, t) = Q(\hat{y}) = 0$, where h_1, h_2 are linear in the variables $\hat{y} = (u, y_0, y_1, y_2, y_3, y_4)$ and $Q(\hat{y})$ is the Lie quadric (for the exact definition see formula (6)). It turns out that the variety defined by the system of equations $h_1(\hat{y}, t) = h_2(\hat{y}, t) = 0$ is a dual variety to the curve $\hat{\mathcal{E}} \in \mathbb{P}^5$, where $\hat{\mathcal{E}}$ is a curve on the Lie quadric determined by the spine curve $\mathcal{E}$ (for the explicit definition see formula (11)). For the dual variety $\mathcal{V}(\hat{\mathcal{E}})$ we define the μ-basis, which consists of two polynomials $p_1(\hat{y}, t), p_2(\hat{y}, t)$ which are linear in $\hat{y}$, and of degree d_1, d_2 in t are such that $d_1 + d_2$ is minimal. It turns out that the resultant of p_1 and p_2 with respect to t gives the implicit equation of the variety $\mathcal{V}(\hat{\mathcal{E}})$. There is a simple substitution formula (see the algorithm at the end of section 5) to compute the implicit equation of the canal surface from the implicit equation of the variety $\mathcal{V}(\hat{\mathcal{E}})$.

Partial solutions to the problem of finding the implicit equation (and degree) for canal surfaces have been given in other papers. For instance, the degree of offsets to curves is studied in [Segundo, Sendra(2005)]. In [Xu et al.(2006)], there is a degree formula for the implicit equation of a polynomial canal surface. Quadratic canal surfaces (parametric and implicit representation) have been studied in [Krasauskas, Zube(2007)].

We close the introduction by noting that the implicit degree of a canal surface is important for the parametric degree. Our observation is that if the canal surface has the minimal parametrization of bi-degree $(2, d)$ then its implicit degree is close to $2d$. On the minimal bi-degree $(2, d)$ parametrizations of the canal surface we refer to [Krasauskas(2007)].

The paper is organized as follows. In the next section we develop some algebraic formalism about modules with two quasi-generators. We define the μ-basis for these modules and present an algorithm for its computation. In the following section, we recall some needed facts about Lie and Laguerre sphere geometry. Then using Lie and Laguerre geometry we describe the canal surface explicitly. Also, we introduce the Γ-hypersurface which contains all d-offsets to the canal surface. Using the μ-basis algorithm we compute the implicit equations of the dual variety $\mathcal{V}(\hat{\mathcal{E}})$, the Γ-hypersurface and the canal surface $\mathcal{C}$. Next we apply the results of the previous section to the dual variety $\mathcal{V}(\hat{\mathcal{E}})$ of the curve and explain how to compute the implicit degree of the Γ-hypersurface (without computation of the implicit equation). Finally, we give some computational examples.

2 Modules with two quasi-generators and the μ-basis.

Let $\mathbb{R}[t]$ be polynomial ring over the field of real numbers, and denote $\mathbb{R}[t]^d$ the R-module of d-dimensional row vectors with entries in $\mathbb{R}[t]$. Let $\mathbb{R}(t)$ be the field of rational functions in t. For a pair of vectors $A = (A_1, A_2, \ldots, A_d)$,$B = (B_1, B_2, ..., B_d) \in \mathbb{R}[t]^d$ the set

$$M = \langle A, B\rangle = \{aA + bB \in \mathbb{R}[t]^d \mid a, b \in \mathbb{R}(t), A, B \in \mathbb{R}[t]^d\} \subset \mathbb{R}[t]^d \quad (1)$$

is the $\mathbb{R}[t]$-module with two polynomial *quasi-generators* A, B. Here, we assume that A, B are $\mathbb{R}[t]$-linearly independent, i.e. $aA + bB = 0$ with $a, b \in \mathbb{R}[t]$ if and only if $a = b = 0$.
Remark: Note that the vectors A, B may not be generators of the module M over $\mathbb{R}[t]$ because a and b in the definition (1) are from the field $\mathbb{R}(t)$ of rational functions. For example, if $A = pD$ with $p \in \mathbb{R}[t], D \in \mathbb{R}[t]^d$ and $\deg p > 0$ then A, B are not generators of the module M.

For $A = (A_1, A_2, \ldots, A_d)$,$B = (B_1, B_2, ..., B_d) \in \mathbb{R}[t]^d$ we define the Plücker coordinate vector $A \wedge B$ as follows:

$$A \wedge B = ([1, 2], [1, 3], ..., [d-1, d]) \in \mathbb{R}[t]^{d(d-1)/2}, \text{ where } [i, j] = A_iB_j - A_jB_i.$$

In other words, $A \wedge B$ is the vector of 2-minors of the matrix

$$W_{A,B} = \begin{pmatrix} A_1 & A_2 & \cdots & A_d \\ B_1 & B_2 & \cdots & B_d \end{pmatrix}$$

and we denote by $\deg(A \wedge B) = \max_{i,j}\{\deg(A_iB_j - A_jB_i)\}$ the degree of the Plücker coordinate vector, i.e. the maximal degree of a 2-minor of $W_{A,B}$.

Let a polynomial vector $A \in \mathbb{R}[t]^d$ be presented as

$$A = \sum_{i=0}^{n} \alpha_i t^i, \ \alpha_i \in \mathbb{R}^d, \ i = 0, ..., n; \ \alpha_n \neq 0.$$

We denote the leading vector α_n by $LV(A)$ and the degree of A by $\deg A = n$.

Note that if $LV(A)$ and $LV(B)$ are linearly independent over $\mathbb{R}$ then $\deg A \wedge B = \deg A + \deg B$ and $LV(A \wedge B) = LV(A) \wedge LV(B)$. We define

$$\deg M = \min\{\deg(\tilde{A} \wedge \tilde{B}) \mid \ \tilde{A}, \tilde{B} \in \mathbb{R}[t]^d \text{ such that } \langle \tilde{A}, \tilde{B}\rangle = M\}$$

to be the degree of the module M with two quasi-generators.

Definition 1. Two quasi-generators $\tilde{A}, \tilde{B}$ of the module $M = \langle A, B\rangle$ are called a μ-basis of the module M if $\deg M = \deg \tilde{A} + \deg \tilde{B}$.

As we always have the inequality $\deg(A \wedge B) \leq \deg A + \deg B$, this means in particular that the sum $\deg \tilde{A} + \deg \tilde{B}$ is minimal. A μ-basis always exists, as we shall see at the end of the section. Let us explain the geometric motivation behind this definition.

Remark 2. By abuse of notation, we will continue to denote parameters t, however in the geometric definitions that follow, they should be understood as parameters $(t : s) \in \mathbb{P}^1$ and polynomials in $\mathbb{R}[t]$ should be thought of as homogenized with respect to a new variable s.

We define the following subspace of $\mathbb{R}^d$ for the module $M = \langle A, B \rangle$.

$$L(M, t_0) = \{x \in \mathbb{R}^d \mid C(t_0) \cdot x = 0 \text{ for all } C \in M\}$$

where $C(t) = (C_1(t), C_2(t), \ldots, C_d(t)) \in \mathbb{R}[t]^d$, $x = (x_1, x_2, \ldots, x_d)^T$ and $C(t) \cdot x = x_1C_1(t) + x_2C_2(t) + \ldots + x_dC_d(t)$. We have the inequality $\dim L(M, t_0) \geq d - 2$, because the module M has only two quasi-generators. In fact, we have $\dim L(M, t_0) = d - 2$ for all t_0, as we will see in Proposition 8.2. Whenever two vectors $A(t_0)$ and $B(t_0)$ are linearly independent in $\mathbb{R}^d$ then $L(M, t_0)$ is the intersection of two hyperspaces $\{x \in \mathbb{R}^d | \, A(t_0) \cdot x = 0\}$ and $\{x \in \mathbb{R}^d \mid B(t_0) \cdot x = 0\}$.

Using those subspaces, we can associate a hypersurface $\mathcal{S}_M$ in the real projective space $\mathbb{P}^{d-1} = \mathbb{P}(\mathbb{R}^d)$ with the module M

$$\mathcal{S}_M := \bigcup_t \mathbb{P}(L(M, t)) \subset \mathbb{P}^{d-1}. \tag{2}$$

Note that this definition and the definition of $L(M, t_0)$ depend only on the module M and not on the choice of quasi-generators. It is useful to compare the hypersurface $\mathcal{S}_M$ with the hypersurface $\mathcal{S}_{A,B}$ defined as

$$\mathcal{S}_{A,B} := \bigcup_t (\{A(t) \cdot x\} \cap \{B(t) \cdot x\}) \subset \mathbb{P}^{d-1} \tag{3}$$

where A, B are quasi-generators of M. By definition, this is the variety defined by $\operatorname{Res}_t(A(t) \cdot x, B(t) \cdot x)$ and it is clear that $\mathcal{S}_M \subset \mathcal{S}_{A,B}$. If the vectors $A(t_0), B(t_0)$ are linearly dependent, then $(\{A(t_0) \cdot x\} \cap \{B(t_0) \cdot x\}) \subset \mathbb{R}^d$ is a subspace of codimension one. Note that in this case the implicit equation $\operatorname{Res}_t(A(t) \cdot x, B(t) \cdot x)$ contains the factor $A(t_0) \cdot x$. As a matter of fact, this happens if and only if $W_{A,B}(t_0)$ has rank one, which is equivalent to saying that t_0 is a zero of the ideal generated by the Plücker coordinates.

In fact, we will see in Proposition 6 that this phenomenon does not occur for μ-bases, i.e. if $\tilde{A}, \tilde{B}$ is a μ-basis of the module M then $\mathcal{S}_M = \mathcal{S}_{\tilde{A},\tilde{B}}$ and there are no extraneous factors as before.

Remark 3. We should explain why we use the term μ-basis. The above definition is a generalization of the usual definition for the μ-basis of a rational ruled surface (as in [Cox, Sederberg, Chen (1998)], [Chen et al.(2001)] or [Dohm(2006)]). They coincide in the special case $d = 4$. M is the analogue of the syzygy module (i.e. the module of moving planes following the parametrization of the ruled surface) and the subspaces $L(M, t)$, which in this case are 2-dimensional and hence define projective lines, are exactly the family of lines which constitute the ruled surface. Similarly, the case $d = 3$ corresponds at the theory of μ-bases for rational curves and our definition is equivalent to the usual definition as in [Chen, Wang(2003), Theorem 3, Condition 3].

However, the approach used here is actually inverse to the approach in the cited papers. In the latter the ruled surface is defined by a parametrization and then the module of moving planes is studied, whereas here we fix a module that "looks like" such a moving plane module and then study the (generalized) ruled surface that corresponds to it. Note that by definition of the subspaces $L(M,t)$ any element C of M can be considered a moving plane following $\mathcal{S}_M$, in the sense that for all $x \in \mathcal{S}_M$ there is a parameter t such that $C(t) \cdot x = 0$.

Note that $A \wedge B$ defines the so-called Plücker curve $\mathcal{P}$ in $\mathbb{P}^{d(d-1)/2-1}$ by

$$\begin{array}{rcc} \varphi_{\mathcal{P}} : \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^{d(d-1)/2-1} \\ t & \mapsto & ([1,2] : [1,3] : \dots : [d-1,d]) \end{array}$$

where $[i,j] = A_i B_j - A_j B_i$. We will denote $k = \deg \varphi_{\mathcal{P}}$ the degree of the parametrization, which is the cardinality of the fiber of a generic point in the image of $\varphi_{\mathcal{P}}$. Note that $\varphi_{\mathcal{P}}$ and k are the same for any choice of quasi-generators of M.

Proposition 4. *For any pair of quasi-generators A, B of M we have the degree formula*

$$k \cdot \deg \mathcal{S}_M = \deg(A \wedge B) - \deg q_{A,B},$$

where $q_{A,B} = \gcd(A \wedge B)$ and $k = \deg \varphi_{\mathcal{P}}$. Moreover, we have $\deg \mathcal{P} = \deg \mathcal{S}_M$.

Proof. The proposition and the proof are similar to Lemma 1 in [Chen et al.(2001)] and to Theorem 5.3 in [Pottmann et al.(1998)].

The implicit degree of the hypersurface $\mathcal{S}_{A,B}$ is the number of intersections between a generic line and the hypersurface. The generic line $L(s)$ is defined by two points in the space $L(s) = H_0 + sH_1$, where $H_i = (h_{i1}, h_{i2}, \dots, h_{id}), i = 0, 1$. The line $L(s)$ intersects the hyperplane $\{A(t) \cdot x\}$ if and only if $H_0 \cdot A(t) + sH_1 \cdot A(t) = 0$. Since the line $L(s)$ should intersect the hyperplane $\{B(t) \cdot x\}$ too, we see that the implicit degree is the number of intersections of two curves in the (t,s) plane:

$$H_0 \cdot A(t) + sH_1 \cdot A(t) = 0,$$

$$H_0 \cdot B(t) + sH_1 \cdot B(t) = 0.$$

Eliminating s from the above equation we have

$$\begin{vmatrix} H_0 \cdot A(t) & H_1 \cdot A(t) \\ H_0 \cdot B(t) & H_1 \cdot B(t) \end{vmatrix} = (H_0 \wedge H_1) \cdot (A(t) \wedge B(t)) = 0, \tag{4}$$

where $C \cdot D$ means a standard scalar product of two vectors $C, D \in \mathbb{R}^{d(d-1)/2}$. The number of solutions of (4) is the number of intersection points of the Plücker curve with a generic hyperplane in $\mathbb{P}^{d(d-1)/2-1}$, so $\deg \mathcal{P} = \deg \mathcal{S}_M$.

Now it is known (see for example [Dohm(2006), Theorem 1]) that

$$k \cdot \deg \mathcal{P} = \deg(A \wedge B) - \deg q_{A,B}$$

and the proposition follows. □

We have yet to show the existence of the μ-basis. To this end, we propose an algorithm for its computation, the basic idea of which is to reduce $q_{A,B} = \gcd(A \wedge B)$ to a constant using the so-called Smith form of the $2 \times d$ matrix

$$W_{A,B} = \begin{pmatrix} A_1 & A_2 & \cdots & A_d \\ B_1 & B_2 & \cdots & B_d \end{pmatrix}$$

and then render the leading vectors linearly independent by a simple degree reduction. The Smith form is a decomposition $W_{A,B} = U \cdot S \cdot V$, with unimodular $U \in \mathbb{R}[t]^{2\times 2}$, $V \in \mathbb{R}[t]^{d\times d}$, and

$$S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q_{A,B} & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}[t]^{2\times d}$$

It always exists and can be computed efficiently by standard computer algebra systems.

Algorithm (μ-basis)

1. INPUT: Quasi-generators $A = (A_1, A_2, \ldots, A_d)$,$B = (B_1, B_2, ..., B_d) \in \mathbb{R}[t]^d$ of the module M
2. Set
$$W_{A,B} = \begin{pmatrix} A_1 & A_2 & \cdots & A_d \\ B_1 & B_2 & \cdots & B_d \end{pmatrix}.$$
3. Compute a Smith form
$$W_{A,B} = U \cdot \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q_{A,B} & 0 & \cdots & 0 \end{pmatrix} \cdot V$$
with unimodular $U \in \mathbb{R}[t]^{2\times 2}$,$V \in \mathbb{R}[t]^{d\times d}$.
4. Set W' to be the $2 \times d$-submatrix consisting of the first two rows of V.
5. If the vector of leading terms (with respect to the variable t) of the first row is h times the one of the second row, $h \in \mathbb{R}[t]$, set $W' := \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \cdot W'$.
6. If the vector of leading terms (with respect to the variable t) of the second row is h times the one of the first row, $h \in \mathbb{R}[t]$, set $W' := \begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} \cdot W'$.
7. If the preceding two steps changed W' go back to Step 5.
8. Set $\tilde{A}, \tilde{B}$ to be the rows of W'.
9. OUTPUT: A μ-basis $\tilde{A}, \tilde{B}$ of the module M

As we shall see in Section 5, the case we are interested in is the case $d = 6$, so we are dealing with very small matrices and the computations are extremely fast. Note that we actually only need the first two rows of V, so we could optimize the algorithm by modifying the Smith form algorithm used as not to compute the unnecessary entries of the matrices U and V. Generally, the number of elementary matrix operations in Step 5 and 6 is very low. In the worst case, it is bounded by the maximal degree of the entries of the matrix W' in Step 4 of the algorithm, since each step reduces the maximal degree in one of the rows of W'.

Next, we will show that the output of the above algorithm is a μ-basis and that the resultant of a μ-basis $\tilde{A}, \tilde{B}$ of the module $M = \langle A, B\rangle$ is an implicit equation of $\mathcal{S}_M$. In Section 5, we will use these results for a special choice of A and B to compute the implicit equation of a canal surface.

Lemma 5. *The output of the above algorithm is a μ-basis and we have $k \cdot \deg \mathcal{S}_M = \deg M$, where $k = \deg \varphi_{\mathcal{P}}$.*

Proof. Let $\tilde{A}(t), \tilde{B}(t)$ be the output of the above algorithm. By construction it is clear that $\tilde{A}(t), \tilde{B}(t)$ are quasi-generators of M and that $\tilde{q}_{A,B} = \gcd(\tilde{A} \wedge \tilde{B}) = 1$. Furthermore, we have $\deg(\tilde{A} \wedge \tilde{B}) = \deg(\tilde{A}) + \deg(\tilde{B})$, because the vectors of leading terms of $\tilde{A}(t)$ and $\tilde{B}(t)$ are linearly independent. So by Proposition 4 we deduce

$$\begin{aligned} k \cdot \deg \mathcal{S}_M &= \deg(\tilde{A} \wedge \tilde{B}) - \deg(\tilde{q}) \\ &= \deg(\tilde{A} \wedge \tilde{B}) \\ &= \deg(\tilde{A}) + \deg(\tilde{B}) \end{aligned}$$

Moreover, by definition we have $\deg M \leq \deg(\tilde{A} \wedge \tilde{B})$ and if A, B are quasi-generators such that $\deg(A \wedge B)$ is minimal, the degree formula gives $\deg(\tilde{A} \wedge \tilde{B}) = \deg(A \wedge B) - \deg(q) \leq \deg M$, which shows that $k \cdot \deg \mathcal{S}_M = \deg M$, and as a consequence that $\tilde{A}(t), \tilde{B}(t)$ is indeed a μ-basis. $\square$

Proposition 6. *Let $\tilde{A}(t), \tilde{B}(t)$ be a μ-basis of M and $x = (x_1, x_2, \ldots, x_d)^T$ variables. Then*

$$\operatorname{Res}_t(\tilde{A}(t) \cdot x, \tilde{B}(t) \cdot x) = F_{\mathcal{S}_M}^k$$

where $F_{\mathcal{S}_M}$ is the implicit equation of the hypersurface $\mathcal{S}_M$.

Proof. First, we will show in the same way as in [Dohm(2006), Theorem 9] that $\operatorname{Res}_t(\tilde{A}(t) \cdot x, \tilde{B}(t) \cdot x)$ is geometrically irreducible, i.e. the power of an irreducible polynomial. As we shall see in Proposition 8, the intersection of the hyperplanes $\{\tilde{A}(t) \cdot x\}$ and $\{\tilde{B}(t) \cdot x\}$ is of codimension 2 for any parameter $t \in \mathbb{P}^1$. So the incidence variety

$$\mathcal{W} = \{(t, x) \in \mathbb{P}^1 \times \mathbb{P}^{d-1} | \tilde{A}(t) \cdot x = \tilde{B}(t) \cdot x = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^{d-1}$$

is a vector bundle over $\mathbb{P}^1$ and hence irreducible. So the projection on $\mathbb{P}^{d-1}$ is irreducible as well and its equation, which is by definition the hypersurface defined by $\operatorname{Res}_t(\tilde{A}(t) \cdot x, \tilde{B}(t) \cdot x)$, is a power of an irreducible polynomial.

As we have remarked earlier, the resultant of two quasi-generators is always a multiple of the implicit equation of $\mathcal{S}_M$, so $\operatorname{Res}_t(\tilde{A}(t) \cdot x, \tilde{B}(t) \cdot x)$ is a power of $F_{\mathcal{S}_M}$.

But using the degree property above we see

$$\deg(\operatorname{Res}_t(\tilde{A}(t) \cdot x, \tilde{B}(t) \cdot x)) = \deg(\tilde{A}) + \deg(\tilde{B}) = k \cdot \deg \mathcal{S}_M$$

which implies that $\operatorname{Res}_t(\tilde{A}(t) \cdot x, \tilde{B}(t) \cdot x)$ equals $F_{\mathcal{S}_M}^k$. $\square$

Remark 7. It is known that the Plücker curve $\mathcal{P}$ can be properly reparametrized, i.e. there exists a rational function h of degree k such that $A \wedge B = C \circ h$, where C is a proper parametrization of $\mathcal{P}$. It is tempting to use this proper reparametrization in order to represent the implicit equation $F_{\mathcal{S}_M}$ of $\mathcal{S}_M$ directly as a resultant as in the proof of [Dohm(2006), Theorem 3]. However, h does not necessarily factorize A and B, i.e. it is not sure that there exist A' and B' with $A = A' \circ h$ and $B = B' \circ h$, which would be needed to do this.

In the following we present some properties of μ-bases. Note that the properties in Propositions 8, 9 are similar to [Chen, Wang(2003)] Theorems 1,3. However, we give different proofs by deducing them from the degree formula and Lemma 5.

Proposition 8. *Let $M = \langle A, B\rangle$ and let $\tilde{A}, \tilde{B}$ be a μ-basis of the module M. Then the following properties hold:*
1. The vectors $LV(\tilde{A}), LV(\tilde{B})$ are linearly independent.
2. $\tilde{A}(t_0), \tilde{B}(t_0)$ are linearly independent over $\mathbb{C}$ for any parameter value $t_0 \in \mathbb{C}$.

Proof. 1. If $LV(\tilde{A}), LV(\tilde{B})$ were linearly dependent, this would imply that $k \cdot \deg \mathcal{S}_M = \deg(\tilde{A} \wedge \tilde{B}) - \deg(q_{\tilde{A},\tilde{B}}) < \deg(\tilde{A}) + \deg(\tilde{B}) = \deg M$ which is a contradiction to Lemma 5.
2. Suppose that $\tilde{A}(t_0), \tilde{B}(t_0)$ are linearly dependent for some $t_0 \in \mathbb{C}$. This is equivalent to saying that the matrix $W_{\tilde{A},\tilde{B}}$ is not of full rank, which means that all 2-minors vanish. So t_0 is a root of $q_{\tilde{A},\tilde{B}}$ and as above we deduce $k \cdot \deg \mathcal{S}_M = \deg(\tilde{A} \wedge \tilde{B}) - \deg(q_{\tilde{A},\tilde{B}}) < \deg(\tilde{A}) + \deg(\tilde{B}) = \deg M$ which is again a contradiction to Lemma 5. ☐

Proposition 9. *Let $M = \langle \tilde{A}, \tilde{B}\rangle$ and assume that $\tilde{A}, \tilde{B}$ satisfy conditions 1,2 from Proposition 8. Then any element $D \in M$ has the following expression: $D = h_1\tilde{A} + h_2\tilde{B}$ for some $h_1, h_2 \in \mathbb{R}[t]$, i.e. $\tilde{A}, \tilde{B}$ are generators of the module M over the polynomial ring $\mathbb{R}[t]$. Moreover, the pair $\tilde{A}, \tilde{B}$ is a μ-basis of the module M.*

Proof. Let $D \in M$, it can be expressed as

$$D = \frac{a}{b}\tilde{A} + \frac{c}{d}\tilde{B}$$

with $a, b, c, d \in \mathbb{R}[t]$ and co-prime numerators and denominators in the rational functions $\frac{a}{b}$ and $\frac{c}{d}$. Furthermore, we may assume that $\gcd(a, c) = 1$, because if $\frac{D}{\gcd(a,c)}$ is a linear combination of $\tilde{A}, \tilde{B}$, then so is D. Multiplying both sides of the above equation with bd we obtain $bdD = ad\tilde{A} + bc\tilde{B}$ or equivalently $b(dD - c\tilde{B}) = ad\tilde{A}$ and since b divides neither a nor $\tilde{A}$ (if it divided $\tilde{A}$, for any root t_0 of b and any constant α we would deduce the relation $0 = \alpha\tilde{A}(t_0) + 0 \cdot \tilde{B}(t_0)$, which contradicts property 2 in Proposition 8), one concludes that b divides d and by a symmetric argument that d divides b, so we may assume $b = d$. So we have

$$bD = a\tilde{A} + c\tilde{B}$$

and plugging a root t_0 of b into the equation, we would obtain a non-trivial linear relation between $\tilde{A}$ and $\tilde{B}$, again a contradiction to Proposition 8. This implies that b and d are constant, which shows that any $D \in M$ can be expressed as linear combination of $\tilde{A}$ and $\tilde{B}$ over $\mathbb{R}[t]$. In other words: $\tilde{A}$ and $\tilde{B}$ are not only quasi-generators of M, but actually generators in the usual sense, i.e. over $\mathbb{R}[t]$.

Suppose that $\deg \tilde{A} \leq \deg \tilde{B}$ and let $M = \langle P_1, P_2\rangle$. Then we proved that $P_i = h_{i1}\tilde{A} + h_{i2}\tilde{B}, i = 1, 2$ for some polynomials $h_{ij} \in \mathbb{R}[t]$. Since $LV(\tilde{A}), LV(\tilde{B})$ are linearly independent $LV(h_{i1}\tilde{A})$ and $LV(h_{i2}\tilde{B}), i = 1, 2$ do not cancel each other. Therefore, $\deg P_1 \geq \deg \tilde{B}$ (if $h_{12} \neq 0$) or $\deg P_2 \geq \deg \tilde{B}$ (if $h_{22} \neq 0$). Also $\deg P_1 \geq \deg \tilde{A}$ and $\deg P_2 \geq \deg \tilde{A}$. So, we see that $\deg P_1 + \deg P_2 \geq \deg \tilde{A} + \deg \tilde{B}$, i.e. a pair $\tilde{A}, \tilde{B}$ is a μ-basis of the module M. ☐

3 Elements of Lie and Laguerre Sphere Geometry

Here we shortly recall the elements of Lie and Laguerre Sphere Geometry (cf. [Cecil(1992), Pottmann, Peternell(1998), Krasauskas, Mäurer(2000)]). We start from the construction of Lie's geometry of oriented spheres and planes in $\mathbb{R}^3$. Let $\mathbf{p} \in \mathbb{R}^3$, $r \in \mathbb{R}$. The oriented sphere $S_{\mathbf{p},r}$ in $\mathbb{R}^3$ is the set

$$S_{\mathbf{p},r} = \{\mathbf{v} \in \mathbb{R}^3 | (\mathbf{v} - \mathbf{p}) \cdot (\mathbf{v} - \mathbf{p}) = r^2\},$$

where by $\mathbf{v} \cdot \mathbf{w}$ we denote the standard positive definite scalar product in $\mathbb{R}^3$. The orientation is determined by the sign of r: the normals are pointing outwards if $r > 0$. If $r = 0$ then $S_{\mathbf{p},0} = \{\mathbf{p}\}$ is a point. Let $\mathbf{n} \in \mathbb{R}^3$ with $\mathbf{n} \cdot \mathbf{n} = 1$ and $h \in \mathbb{R}$. The oriented plane $P_{\mathbf{n},h}$ in $\mathbb{R}^3$ is the set

$$P_{\mathbf{n},h} = \{\mathbf{v} \in \mathbb{R}^3 | \mathbf{v} \cdot \mathbf{n} = h\}.$$

The *Lie scalar* product with signature $(4, 2)$ in $\mathbb{R}^6$ is defined by the formula

$$[x, z] = \frac{-x_1 z_2 - x_2 z_1}{2} + x_3 z_3 + x_4 z_4 + x_5 z_5 - x_6 z_6.$$

for $x = (x_1, \ldots, x_6)$ and $z = (z_1, \ldots, z_6)$. In matrix notation we have

$$[x, z] = xCz^T, \quad \text{where } xC = (-x_2/2, -x_1/2, x_3, x_4, x_5, -x_6). \tag{5}$$

Denote $\hat{y} = (u : y_0 : y_1 : y_2 : y_3 : y_4) \in \mathbb{P}(\mathbb{R}^6) = \mathbb{P}^5$ and define the quadric

$$\mathcal{Q} = \{\hat{y} \in \mathbb{P}^5 \mid [\hat{y}, \hat{y}] = -uy_0 + y_1^2 + y_2^2 + y_3^2 - y_4^2 = 0\} \tag{6}$$

where $[.,.]$ is the obvious extension of the Lie scalar product to $\mathbb{P}^5$. $\mathcal{Q}$ is called *Lie quadric*.

We represent an oriented sphere $S_{\mathbf{p},r}$ (or an oriented plane $P_{\mathbf{n},h}$) as a point $Lie(S_{\mathbf{p},r})$ (resp. $Lie(P_{\mathbf{n},h})$) on the Lie quadric:

$$\begin{aligned} Lie(S_{\mathbf{p},r}) &= (2(\mathbf{p} \cdot \mathbf{p} - r^2), 2, 2\mathbf{p}, 2r) \in \mathcal{Q}, \quad \mathbf{p} \in \mathbb{R}^3, r \in \mathbb{R}, \\ Lie(P_{\mathbf{n},h}) &= (2h, 0, \mathbf{n}, 1) \in \mathcal{Q}, \quad \mathbf{n} \in \mathbb{R}^3,\ h \in \mathbb{R}. \end{aligned}$$

It is easy to see that we have determined a bijective correspondence between the set of points on the Lie quadric $\mathcal{Q}$ and the set of all oriented spheres/planes in $\mathbb{R}^3$. Here we assume that a point $q = (1 : 0 : 0 : 0 : 0 : 0) \in \mathcal{Q}$ on the Lie quadric $\mathcal{Q}$ corresponds to an infinity, i.e. to a point in the compactification of $\mathbb{R}^3$. We say that $q = (1 : 0 : 0 : 0 : 0 : 0)$ is the improper point on the Lie quadric. Notice that oriented planes in $\mathbb{R}^3$ correspond to points $\mathcal{Q} \cap T_q$, where $T_q = \{\hat{y} = (u : y_0 : y_1 : y_2 : y_3 : y_4) \in \mathbb{P}^5 | y_0 = 0\}$ is a tangent hyperplane to the Lie quadric at the improper point q.

Two oriented spheres $S_{\mathbf{p}_1,r_1}, S_{\mathbf{p}_2,r_2}$ are in *oriented contact* if they are tangent and have the same orientation at the point of contact. The analytic condition for oriented contact is

$$\|\mathbf{p}_1 - \mathbf{p}_2\| = |r_1 - r_2|,$$

where $\|\mathbf{p}_1 - \mathbf{p}_2\|$ denotes the usual distance between two points in the Euclidean space $\mathbb{R}^3$. One can check directly that the analytical condition of oriented contact on the Lie quadric is equivalent to the equation

$$[Lie(S_{\mathbf{p}_1,r_1}), Lie(S_{\mathbf{p}_2,r_2})] = 0.$$

It is known that the Lie quadric contains projective lines but no linear subspaces of higher dimension (Chapter 1, Corollary 5.2 in [Cecil(1992)]). Moreover, the line in $\mathbb{P}^5$ determined by two points k_1, k_2 of $\mathcal{Q}$ lies on $\mathcal{Q}$ if and only $[k_1, k_2] = 0$, i.e. the corresponding spheres to k_1, k_2 are in an oriented contact (Chapter 1, Theorem 1.5.4 in [Cecil(1992)]). The points on a line on $\mathcal{Q}$ form so called *parabolic pencil* of spheres. All spheres which correspond to a line on $\mathcal{Q}$ are precisely the set of all spheres in an oriented contact.

Remark 10. Here we use a slightly different coordinate system in Lie Geometry than in the book [Cecil(1992)]. The scalar product as in [Cecil(1992)] may be obtained applying the following transformation:
$x'_1 = (x_1 + x_2)/2, x'_2 = (x_2 - x_1)/2, x'_3 = x_3, x'_4 = x_4, x'_5 = x_5, x'_6 = x_6.$

We show now that the set of points $\hat{y}$ in $\mathcal{Q}$ with $y_0 \neq 0$ is naturally diffeomorphic to the affine space $\mathbb{R}^4$. This diffeomorphism is defined by the map

$$\begin{array}{cccc} \phi : & \mathcal{Q} \setminus T_q & \to & \mathbb{R}^4, \\ & (u : y_0 : y_1 : y_2 : y_3 : y_4) & \mapsto & \left(\frac{y_1}{y_0}, \frac{y_2}{y_0}, \frac{y_3}{y_0}, \frac{y_4}{y_0}\right), \end{array}$$

where $T_q = \{\hat{y} = (u : y_0 : y_1 : y_2 : y_3 : y_4) \in \mathbb{P}^5 \mid y_0 = 0\}$ as before, i.e. the tangent hyperplane to the Lie quadric $\mathcal{Q}$ at the improper point $q = (1 : 0 : 0 : 0 : 0 : 0)$. Let $v = (v_1, v_2, v_3, v_4), w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$ and denote by

$$\langle v, w \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4$$

the Lorentz scalar product on $\mathbb{R}^4$, which can be seen as the restriction of the Lie scalar product $[.,.]$ to $\mathbb{R}^4$. The affine space $\mathbb{R}^4$ with the Lorentz scalar product is called the Lorentz space and denoted by $\mathbb{R}^4_1$.

Let $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. One can check that inverse map of ϕ is given by the formula:

$$\phi^{-1}(y) = (\langle y, y \rangle, 1, y) \in \mathcal{Q} \setminus T_q$$

Notice, that $\phi(Lie(S_{\mathbf{p},r})) = (\mathbf{p}, r)$, i.e. the sphere $S_{\mathbf{p},r} \in \mathbb{R}^3$ corresponds to a point $(\mathbf{p}, r) \in \mathbb{R}^4_1$. The map ϕ can be extended to a linear projection Φ from $\mathcal{Q} \setminus \{q\}$ to $\mathbb{P}^4$ defined as

$$\begin{array}{cccc} \Phi : & \mathcal{Q} \setminus \{q\} & \to & \mathbb{P}^4 \\ & (u : y_0 : y_1 : y_2 : y_3 : y_4) & \mapsto & (y_0 : y_1 : y_2 : y_3 : y_4) \end{array}$$

The points of $\mathcal{Q} \cap T_q$ can be represented as $Lie(P_{\mathbf{n},h}) = (2h, 0, \mathbf{n}, 1)$ and these points correspond to planes in $\mathbb{R}^3$. Note that

$$\Phi(Lie(P_{\mathbf{n},h})) = (0, \mathbf{n}, 1) \in \Omega = \{y_0 = 0, y_1^2 + y_2^2 + y_3^2 - y_4^2 = 0\}$$

are infinite points to the natural extension of $\mathbb{R}^4$ to $\mathbb{P}^4$ which correspond to a pencil of parallel planes in $\mathbb{R}^3$. The quadric Ω is called *absolute quadric*. The preimage of the map Φ has the following form

$$\Phi^{-1}(\overline{y}) = (\langle y, y \rangle : y_0^2 : y_0 y_1 : y_0 y_2 : y_0 y_3 : y_0 y_4) \in \mathcal{Q} \setminus \{q\} \tag{7}$$

where $\overline{y} = (y_0 : y_1 : y_2 : y_3 : y_4) \in \mathbb{P}^4$ and $y = (y_1, y_2, y_3, y_4)$ as before.

A direct computation shows that for $v, w \in \mathbb{R}^4$

$$-2[\phi^{-1}(v), \phi^{-1}(w)] = \langle v - w, v - w \rangle \tag{8}$$

The formula shows that two oriented spheres defined by v, w (i.e. spheres $S_{(v_1,v_2,v_3),v_4}$ and $S_{(w_1,w_2,w_3),w_4}$) are in oriented contact if and only if $\langle v-w, v-w\rangle = 0$.

Let us define two maps: an embedding $i_d : \mathbb{R}^3 \to \mathbb{R}^4, i_d(\mathbf{p}) = (\mathbf{p}, d), d \in \mathbb{R}$ and a projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3, \pi(\mathbf{p}, r) = \mathbf{p}$, where $r \in \mathbb{R}$. We will treat points $i_0(\mathbb{R}^3)$ as spheres with zero radius and identify them with $\mathbb{R}^3$. All interrelations between the spaces introduced above can be described in the following diagram

$$\begin{array}{ccccccc} & & \mathcal{Q} \setminus T_q & \subset & \mathcal{Q} \setminus \{q\} & \subset & \mathbb{P}^5 \\ & & \downarrow \phi & & \downarrow \Phi & & \\ \mathbb{R}^3 & \xrightarrow{i_d} & \mathbb{R}^4 & \subset & \mathbb{P}^4 & & \\ \| & & \downarrow \pi & & & & \\ \mathbb{R}^3 & = & \mathbb{R}^3 & & & & \end{array} \tag{9}$$

Definition 11. For an oriented surface (curve or point) $\mathcal{M} \subset \mathbb{R}^3$ define an *isotropic hypersurface* $\mathcal{G}(\mathcal{M}) \subset \mathbb{P}^4$ as the union of all points in $\mathbb{R}^4$ which correspond to oriented tangent spheres of $\mathcal{M}$. Let $\mathcal{G}_d(\mathcal{M}) = \mathcal{G}(\mathcal{M}) \cap \{y_4 = dy_0\}$ be a variety which corresponds to tangent spheres with radius d of $\mathcal{M}$. The set $\mathrm{Env}_d(\mathcal{M}) = \pi(\mathcal{G}_d(\mathcal{M})|_{\mathbb{R}^4}) \subset \mathbb{R}^3$ are centers of spheres with radius d tangent to $\mathcal{M}$. The set $\mathrm{Env}_d(\mathcal{M})$ is called d-envelope of the variety $\mathcal{M}$. Since $\mathcal{G}(\mathcal{M}) = \bigcup_d \mathcal{G}_d(\mathcal{M})$ we can treat the isotropic hypersurface $\mathcal{G}(\mathcal{M})$ as the union of all d-envelope to the variety $\mathcal{M}$.

If $y, a \in \mathbb{R}^4,\ a_0, y_0 \in \mathbb{R}$, we define a function

$$\begin{aligned} g((a_0 : a), (y_0 : y)) &= \langle a y_0 - a_0 y, a y_0 - a_0 y \rangle = y_0^2 a_0^2 \left\langle \frac{a}{a_0} - \frac{y}{y_0}, \frac{a}{a_0} - \frac{y}{y_0} \right\rangle \\ &= y_0^2 \langle a, a \rangle - 2 a_0 y_0 \langle a, y \rangle + a_0^2 \langle y, y \rangle. \end{aligned} \tag{10}$$

Let $(y_0 : y)$ be such that $g((a_0 : a), (y_0 : y)) = 0$. By the formula (8) we see that spheres $S_{\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0}\right), \frac{a_4}{a_0}}$ and $S_{\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}, \frac{y_3}{y_0}\right), \frac{y_4}{y_0}}$ are in oriented contact. Therefore, in the same manner as previously, we define the isotropic hypersurface $\mathcal{G}((a_0 : a))$ as follows

$$\mathcal{G}((a_0 : a)) = \{(y_0, y) \in \mathbb{P}^4 \mid g((a_0 : a), (y_0 : y)) = 0\} \subset \mathbb{P}^4.$$

In fact, $\mathcal{G}((a_0 : a))$ is a quadratic cone with a singular point at a vertex $(a_0 : a) \in \mathbb{P}^4$ and may be viewed as the set of all spheres which touches the fixed sphere $S_{\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0}\right), \frac{a_4}{a_0}}$. After the restriction to the linear subspace $y_4 = dy_0$ this hypersurface consists of all spheres with radius d which are in oriented contact with the sphere $S_{\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0}\right), \frac{a_4}{a_0}}$ which we denote as $\mathcal{G}_d((a_0 : a)) = \mathcal{G}((a_0 : a)) \cap \{y_4 = dy_0\}$. We notice that $\mathcal{G}_d((a_0 : a))|_{y_0=1}$ is defined by the equation $(a_1 - a_0 y_1)^2 + (a_2 - a_0 y_2)^2 + (a_3 - a_0 y_3)^2 = (a_4 - a_0 d)^2$, i.e.

$$\begin{aligned} \pi(\mathcal{G}_d((a_0 : a))|_{\mathbb{R}^4}) &= S_{\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0}\right), \frac{a_4 - a_0 d}{a_0}} = \mathrm{Env}_{-d}\left(S_{\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0}\right), \frac{a_4}{a_0}}\right) \quad \text{and} \\ \mathcal{G}_d((a_0 : a))|_{\mathbb{R}^4} &= i_d\left(S_{\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0}\right), \frac{a_4 - a_0 d}{a_0}}\right) \end{aligned}$$

Therefore, in this case, the isotropic hypersurface $\mathcal{G}((a_0 : a))$ may be treated as a union all envelopes to the sphere $S_{\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0}\right), \frac{a_4}{a_0}}$. In the next section we generalize the definition of the isotropic hypersurface $\mathcal{G}(\mathcal{M})$ for a curve $\mathcal{M}$ in $\mathbb{R}^4$ (or $\mathbb{P}^4$).

All lines in $\mathbb{R}^4_1$ with directional vectors v can be classified into three types depending on the sign of $\langle v, v\rangle$: (+)-lines, (0)-lines (also called *isotropic* lines), and (−)-lines.

4 The isotropic hypersurface and d-envelopes

In this section, we will see that the definition of the canal surface is not obvious and and we will introduce some geometrical object related to it. A canal surface is given by a so-called *spine curve* $\mathcal{E}$, which is the closed image (with respect to the Zariski topology) of a rational map

$$\begin{array}{ccc} \mathbb{R} & \dashrightarrow & \mathbb{R}^4 \\ t & \mapsto & \left(\frac{e_1(t)}{e_0(t)}, \frac{e_2(t)}{e_0(t)}, \frac{e_3(t)}{e_0(t)}, \frac{e_4(t)}{e_0(t)}\right) \end{array}$$

with polynomials $e_0(t), \ldots, e_4(t) \in \mathbb{R}[t]$ such that $n = \max_{i=0,.,4}\{\deg(e_i(t))\}$. For abbreviation, we usually skip the variable t in the notations. The spine curve describes a family of spheres $\{S_{\left(\frac{e_1(t)}{e_0(t)}, \frac{e_2(t)}{e_0(t)}, \frac{e_3(t)}{e_0(t)}\right), \frac{e_4(t)}{e_0(t)}} \mid t \in \mathbb{R}\}$ whose centers are given by the first three coordinates $\left(\frac{e_1(t)}{e_0(t)}, \frac{e_2(t)}{e_0(t)}, \frac{e_3(t)}{e_0(t)}\right)$ and whose radii are given by the last coordinate $\frac{e_4(t)}{e_0(t)}$. Intuitively, the canal surface is the envelope of this family of spheres, but there are some subtleties to consider before we can make a precise definition.

We can also consider the spine curve as a projective curve $\overline{\mathcal{E}}$ given as the closed image of a parametrization

$$\begin{array}{ccc} \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^4 \\ t & \mapsto & (e_0(t) : e_1(t) : e_2(t) : e_3(t) : e_4(t)) \end{array}$$

with the non-restrictive condition $\gcd(e_0, \ldots, e_4) = 1$, which means that there are no base-points (i.e. parameters for which the map is not well-defined).

Note that in this case the polynomials e_i are actually to be considered as homogenized to the same degree n with respect to a new variable s. As there is a one-to-one correspondence between the univariate polynomials of a certain degree and their homogeneous counterparts, we will keep the notation from above and distinguish between the affine and projective case only where it is necessary to avoid confusion.

In the following we use the notations

$$e = (e_1, e_2, e_3, e_4),\ y = (y_1, y_2, y_3, y_4),$$

$$\overline{e} = (e_0 : e_1 : e_2 : e_3 : e_4),\ \overline{y} = (y_0 : y_1 : y_2 : y_3 : y_4).$$

We first proceed to define a hypersurface in $\mathbb{P}^4$ which is closely related to the canal surface.

Definition 12. The *isotropic hypersurface* $\mathcal{G}(\overline{\mathcal{E}}) = \{\overline{y} \mid G(\overline{y}) = 0\} \subset \mathbb{P}^4$ associated with the (projective) spine curve $\overline{\mathcal{E}}$ is the variety in $\mathbb{P}^4$ defined by the polynomial $G(\overline{y}) = \operatorname{Res}_t(g_1, g_2)$ where

$$\begin{aligned} g_1(\overline{y}, t) &= g(\overline{e}, \overline{y}) = (e_0y_1 - e_1y_0)^2 + (e_0y_2 - e_2y_0)^2 + \\ &\qquad (e_0y_3 - e_3y_0)^2 - (e_0y_4 - e_4y_0)^2 \\ &= \langle e_0y - y_0e, e_0y - y_0e\rangle = e_0^2\langle y, y\rangle - 2\langle e_0e, y_0y\rangle + y_0^2\langle e, e\rangle, \\ g_2(\overline{y}, t) &= \frac{\partial g_1(\overline{y}, t)}{\partial t} = 2(e_0e_0'\langle y, y\rangle - \langle (e_0e)', y_0y\rangle + y_0^2\langle e', e\rangle). \end{aligned}$$

So, we define $\mathcal{G}(\overline{\mathcal{E}})$ as the envelope of the family of isotropic hypersurfaces $\mathcal{G}(\overline{e}) = \mathcal{G}((e_0(t) : e(t)))$.

In the previous section we showed that $\mathcal{G}_d(\overline{e})|_{\mathbb{R}^4} = \operatorname{Env}_{-d}\left(S_{\left(\frac{e_1}{e_0}, \frac{e_2}{e_0}, \frac{e_3}{e_0}\right), \frac{e_4}{e_0}}\right)$. This interpretation leads to the following definition.

Definition 13. The *d-envelope* associated with the (projective) spine curve $\overline{\mathcal{E}}$ is defined as the hypersurface $\operatorname{Env}_d(\overline{\mathcal{E}}) \subset \mathbb{P}^3$ given by the implicit equation

$$G_d(y_0, y_1, y_2, y_3) = \operatorname{Res}_t(g_1|_{y_4=-dy_0}, g_2|_{y_4=-dy_0}) = \operatorname{Res}_t(g_1, g_2)|_{y_4=-dy_0},$$

i.e. the equation obtained by replacing y_4 in $G(\overline{y})$ by $-dy_0$, where $d \in \mathbb{R}$.

The *affine envelope* $\operatorname{Env}_d(\mathcal{E})$ at distance d is the restriction of $\operatorname{Env}_d(\overline{\mathcal{E}})$ to the affine space $\mathbb{R}^3$, defined by the equation $G_d|_{y_0=1} = \operatorname{Res}_t(g_1, g_2)|_{y_4=-dy_0, y_0=1}$, i.e. by setting $y_0 = 1$.

So $\mathcal{G}(\overline{\mathcal{E}})$ contains all offsets associated with the spine curve $\overline{\mathcal{E}}$. Indeed, the surface

$$\operatorname{Env}_d(\overline{\mathcal{E}}) = \mathcal{G}(\overline{\mathcal{E}}) \cap \{y_4 = -dy_0\}$$

is a hyperplane section of $\mathcal{G}(\overline{\mathcal{E}})$, which can be interpreted as a parametrization of all offsets (with respect to the parameter y_4).

The special case $d = 0$ is particularly important. For the real part of $\operatorname{Env}_0(\mathcal{E})$ to be non-empty, one has to suppose that $\mathcal{E}$ has tangent (+)-lines almost everywhere, or equivalently that $\langle e, e\rangle > 0$ almost everywhere. $\operatorname{Env}_0(\mathcal{E})$ is the envelope of the family of spheres in $\mathbb{R}^3$ given by the spine curve $\mathcal{E}$ and $\operatorname{Env}_d(\mathcal{E})$ is the envelope of the same family of spheres with radii augmented by d. For instance, circular cylinders or circular cones (call them just *cones*) are envelopes $\operatorname{Env}_0(\mathcal{L})$ of (+)-lines $\mathcal{L}$ and vice versa. In the literature, the canal surface $\mathcal{C}$ is usually defined as this envelope $\operatorname{Env}_0(\mathcal{E})$. However, we will show in an example that these envelopes can contain "unwanted" extraneous factors, which are geometrically counterintuitive.

Example 14. Consider the spine curve $\mathcal{E}$ given by

$$\left(\frac{e_1(t)}{e_0(t)}, \frac{e_2(t)}{e_0(t)}, \frac{e_3(t)}{e_0(t)}, \frac{e_4(t)}{e_0(t)}\right) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, 0, \frac{1}{2}\right).$$

The first three coordinates describe a circle in the plane and moving spheres of constant radius along this curve, so intuitively the envelope should be a torus

$\mathcal{T}$. But it turns out that the implicit equation of $\mathrm{Env}_0(\mathcal{E})$ is up to a constant computed as

$$G_0 = \mathrm{Res}_t(g_1, g_2)|_{y_4=0,y_0=1} = (y_1^2 + y_2^2)^2(4y_1^2 + 4y_2^2 + 4y_3^2 + 8y_1 + 3)F_{\mathcal{T}}$$

where $F_{\mathcal{T}}$ is indeed the equation of the torus. To understand where the other factors come from, consider the following: For a given parameter t, the equations g_1 and g_2 define spheres $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ in $\mathbb{R}^3$ and

$$\mathrm{Env}_0(\mathcal{E}) = \bigcup_t \mathcal{S}_1(t) \cap \mathcal{S}_2(t)$$

of the intersections of these spheres (actually this is nothing else than the geometric definition of the resultant). Now, while for almost all t this intersection is a transversal circle on the torus (often called characteristic circle in the literature), it can happen that the spheres degenerate either to planes or to the whole space. In our example, for the parameters $t = i$ and $t = -i$ we have $g_1(i) = g_1(-i) = 0$, $g_2(i) = -iy_1 + y_2$ and $g_2(-i) = iy_1 + y_2$, so the intersection in those parameters actually degenerates to (complex) planes which correspond to the factor $(-iy_1 + y_2)(iy_1 + y_2) = y_1^2 + y_2^2$. In the parameter $t = \infty$, both g_1 and g_2 define the same sphere whose equation $4y_1^2 + 4y_2^2 + 4y_3^2 + 8y_1 + 3$ is the other extraneous factor. This kind of phenomenon can also happen for real parameter values, but it is interesting to remark that even though we consider a real parametrization, non-real parameters can interfere with the envelope, because the resultant "knows" about them.

This example shows that $\mathrm{Env}_0(\mathcal{E})$ is not a suitable definition for the canal surface $\mathcal{C}$ and we will later develop one that avoids the kind of extraneous components we have observed.

Remark 15. Sometimes, in the literature, $\mathrm{Env}_0(\mathcal{E})$ is defined in affine space as the resultant

$$\begin{aligned}
\check{G}_0(y_1, y_2, y_3) &= \mathrm{Res}_t(\check{g}_1(y_1, y_2, y_3, t), \check{g}_2(y_1, y_2, y_3, t)), \text{ where} \\
\check{g}_1 &= e_0^2 \check{f}_1, \qquad \check{g}_2 = e_0^3 \check{f}_2, \\
\check{f}_1 &= \left(y_1 - \frac{e_1}{e_0}\right)^2 + \left(y_2 - \frac{e_2}{e_0}\right)^2 + \left(y_3 - \frac{e_3}{e_0}\right)^2 - \left(\frac{e_4}{e_0}\right)^2, \\
\check{f}_2 &= \frac{\partial \hat{f}_1}{\partial t}
\end{aligned}$$

or in other words by deriving the affine equation of the sphere after the substitutions $y_4 = 0, y_0 = 1$ and homogenizing afterwards. Note that in this case $\check{f}_2$ and $\check{g}_2$ are linear in y_1, y_2, y_3. Let $\tilde{f}_1 = g_1|_{y_4=0,y_0=1}$ and $\tilde{f}_2 = g_2|_{y_4=0,y_0=1}$. An easy computation shows that we have the following equalities

$$\check{g}_1 = \tilde{f}_1, \quad \check{g}_2 = e_0\tilde{f}_2 - 2e_0'\tilde{f}_1.$$

Therefore, by properties of the resultant (29),(30), we have

$$\begin{aligned}
\mathrm{Res}_t(\check{g}_1, \check{g}_2) &= \mathrm{Res}_t(\tilde{f}_1, e_0\tilde{f}_2 - 2e_0'\tilde{f}_1) \\
&= \mathrm{Res}_t(\tilde{f}_1, e_0\tilde{f}_2) \\
&= \mathrm{Res}_t(\tilde{f}_1, e_0) \cdot \mathrm{Res}_t(\tilde{f}_1, \tilde{f}_2).
\end{aligned}$$

Hence, we have $\check{G}_0 = \mathrm{Res}_t(\tilde{f}_1, e_0) \cdot G_0$, so there are even more extraneous factors than before due to the roots of e_0.

Linearizing the problem

The main idea to understand and eliminate the extraneous components that appeared in the example is to linearize the equations g_1 and g_2 by replacing the quadratic term $\langle y, y\rangle$ by a new variable u (or more precisely uy_0 to keep the equations homogeneous). This will make the results developed in Section 2 applicable. Geometrically, this means that we will pull back the spine curve to $\mathcal{Q}$ via the correspondence Φ.

For a spine curve $\mathcal{E} \in \mathbb{R}^4_1$ we define a *proper* pre-image $\hat{\mathcal{E}}$ in the Lie quadric $\mathcal{Q}$ as the closure of the set $\hat{\mathcal{E}} = \Phi^{-1}(\mathcal{E})$ in $\mathcal{Q}$. It is immediate by (7) that the parametrization of $\hat{\mathcal{E}}$ is

$$\begin{array}{lll} \mathbb{P}^1 & \dashrightarrow & \mathcal{Q} \subset \mathbb{P}^5 \\ t & \mapsto & (\langle e(t), e(t)\rangle : e_0^2(t) : e_0(t)e_1(t) : e_0(t)e_2(t) : e_0(t)e_3(t) : e_0(t)e_4(t)) \end{array} \tag{11}$$

We can now define the envelopes associated with this new spine curve as follows.

Definition 16. The variety $\mathcal{H}(\hat{\mathcal{E}}) \subset \mathbb{P}^5$ associated with $\hat{\mathcal{E}}$ is the hypersurface in $\mathbb{P}^5$ defined by the implicit equation $H(\hat{y}) = \mathrm{Res}_t(h_1, h_2)$ where $\hat{y} = (u : y_0 : y_1 : y_2 : y_3 : y_4)$ and

$$\begin{aligned} h_1(\hat{y}, t) &= -2[\hat{y}, \hat{\mathcal{E}}(t)] = ue_0^2 + y_0\langle e, e\rangle - 2\langle e_0 e, y\rangle, \\ h_2(\hat{y}, t) &= \frac{\partial h_1(\hat{y}, t)}{\partial t} = -2[\hat{y}, \hat{\mathcal{E}}'(t)] = 2(ue_0e_0' + y_0\langle e', e\rangle - \langle (e_0e)', y\rangle). \end{aligned}$$

Similarly, the variety $\mathcal{H}_d(\hat{\mathcal{E}}) \subset \mathbb{P}^4$ is defined by the implicit equation

$$H_d(u, y_0, y_1, y_2, y_3) = \mathrm{Res}_t(h_1|_{y_4=-dy_0}, h_2|_{y_4=-dy_0}) = \mathrm{Res}_t(h_1, h_2)|_{y_4=-dy_0},$$

i.e. the equation obtained by replacing y_4 in $H(\overline{y})$ by $-dy_0$, where $d \in \mathbb{R}$.

Of course this is nothing else than substituting $\langle y, y\rangle$ in g_1 and g_2 by uy_0 and dividing by y_0, so $g_i(\overline{y}) = h_i(\langle y, y\rangle, y_0^2, y_0 y), i = 1, 2$, i.e. $g_i = h_i \circ \Phi^{-1}, i = 1, 2$. Now as an immediate corollary we obtain

Proposition 17. *With the notations as above we have*

$$G(\overline{y}) = H(\langle y, y\rangle, y_0^2, y_0 y), \ \textit{i.e.}\ G = H \circ \Phi^{-1}, \tag{12}$$

and

$$G_d(y_0, y_1, y_2, y_3) = H_d(y_1^2 + y_2^2 + y_3^2 - d^2y_0^2, y_0^2, y_0y_1, y_0y_2, y_0y_3). \tag{13}$$

To sum up, we have defined two hypersurfaces as resultants of two quadratic forms: $\mathrm{Env}_d(\overline{\mathcal{E}}) \subset \mathbb{P}^3$, which are the offsets to the spine curve $\overline{\mathcal{E}}$, and $\mathcal{G}(\overline{\mathcal{E}}) \subset \mathbb{P}^4$, which can be interpreted as a parametrization of those offsets. As seen in an example, these definitions can lead to additional components which are against the geometric intuition, so it is desirable to give another definition which avoids those extra factors. To this end, we have linearized the problem by replacing the quadratic polynomials g_1 and g_2 by linear forms h_1 and h_2 by substituting the quadratic term by a new variable and have seen how to reverse this substitution. Geometrically, this means that we replace the hypersurfaces $\mathrm{Env}_d(\overline{\mathcal{E}})$ and $\mathcal{G}(\overline{\mathcal{E}})$ by hypersurfaces $\mathcal{H}_d(\hat{\mathcal{E}})$ and $\mathcal{H}(\hat{\mathcal{E}})$ in one dimension higher.

This has the advantage that we can now apply the technique of μ-bases developed earlier to understand and eliminate the extraneous factors of $\mathcal{H}_d(\hat{\mathcal{E}})$ and $\mathcal{H}(\hat{\mathcal{E}})$ and then come back to $\mathbb{P}^3$ (resp. $\mathbb{P}^4$) with the substitution formulae of Proposition 17.

5 The dual variety, offsets, and the canal surface.

In this section, we will finally be able to define the canal surface $\mathcal{C}$ (and more general offsets to it) and the so-called dual variety $\Gamma(\overline{\mathcal{E}})$, which can be seen as a parametrization of the offsets to $\mathcal{C}$.

Up to the constant -2 the system $h_1 = h_2 = 0$ is equal to

$$\begin{cases} \left[\hat{y}, \hat{\mathcal{E}}(t)\right] = \hat{\mathcal{E}}(t) C \hat{y}^T = 0, \\ \left[\hat{y}, \hat{\mathcal{E}}'(t)\right] = \hat{\mathcal{E}}'(t) C \hat{y}^T = 0, \end{cases} \tag{14}$$

where the matrix C is defined by the formula (5).

We can interpret the variety $\mathcal{H}(\hat{\mathcal{E}})$ defined by (14) as a dual variety to the curve $\hat{\mathcal{E}}$ with respect to the Lie quadric $\mathcal{Q}$, i.e. the dual variety to the curve $\hat{\mathcal{E}}(t)C$. Indeed, this dual variety consists of the hyperplanes which touch the curve $\hat{\mathcal{E}}(t)C$. The first equation in (14) means that the hyperplane contains the point $\hat{\mathcal{E}}(t)C$, the second equation means that the hyperplane contains the tangent vector $\hat{\mathcal{E}}'(t)C$ to the curve $\hat{\mathcal{E}}(t)C$.

In order to simplify notation we denote

$$E = \hat{\mathcal{E}}(t)C = \left(-\frac{e_0^2}{2}, -\frac{\langle e, e\rangle}{2}, e_0e_1, e_0e_2, e_0e_3, -e_0e_4\right) \tag{15}$$

$$\begin{aligned} E' = \hat{\mathcal{E}}'(t)C = (&-e_0e_0', -\langle e', e\rangle, e_0'e_1 + e_0e_1', e_0'e_2 + e_0e_2', \\ &e_0'e_3 + e_0e_3', -e_0'e_4 - e_0e_4') \end{aligned} \tag{16}$$

and we have that $\mathcal{H}(\hat{\mathcal{E}}) = \mathcal{S}_{E,E'}$ by (3). As we have seen in Section 2, this surface contains extraneous factors which correspond to the roots of the 2-minors of the matrix $W_{E,E'}$, but which can be eliminated by replacing E, E' by a μ-basis of the module $\langle E, E'\rangle$. It is thus natural to make the following definition.

Definition 18. We define the dual variety $\mathcal{V}(\hat{\mathcal{E}}) \subset \mathbb{P}^5$ to the curve $\hat{\mathcal{E}}$ as the hypersurface

$$\mathcal{V}(\hat{\mathcal{E}}) = \mathcal{S}_{\langle E,E'\rangle} \tag{17}$$

where $\langle E, E'\rangle$ is the module quasi-generated by E and E'.

By the results of Section 2, it is immediate that $\mathcal{V}(\hat{\mathcal{E}}) \subset \mathcal{H}(\hat{\mathcal{E}})$ does not contain the components of $\mathcal{H}(\hat{\mathcal{E}})$ caused by parameters t where $W_{E,E'}(t)$ is not of full rank or equivalently, where the intersection of the hyperplanes defined by h_1 and h_2 is of codimension 1, i.e. the hyperplanes coincide. So we can deduce

Proposition 19. *Let E_1, E_2 be a μ-basis of the module quasi-generated by E and E' and let k be the degree of the parametrization $E \wedge E'$ as in Section 2. Then*

$$k \cdot \deg \mathcal{V}(\hat{\mathcal{E}}) = \deg E_1 + \deg E_2 = \deg(E \wedge E') - \deg q_{E,E'},$$

where $q_{E,E'} = \gcd(E \wedge E')$ and

$$\mathrm{Res}_t(E_1 \cdot \hat{y}^T, E_2 \cdot \hat{y}^T) = F^k_{\mathcal{V}(\hat{\mathcal{E}})},$$

where $F_{\mathcal{V}(\hat{\mathcal{E}})}$ is the implicit equation of $\mathcal{V}(\hat{\mathcal{E}})$.

Proof. It follows directly from Propositions 4 and 6. $\square$

Of course, the same considerations can be applied to the hypersurfaces $\mathcal{H}_d(\hat{\mathcal{E}})$ and we make the analogous definitions. Substituting $y_4 = -dy_0$ in h_1 and h_2 corresponds to replacing E and E' by two linear forms

$$\begin{aligned} D &= \left(-\frac{e_0^2}{2}, -\frac{\langle e,e\rangle}{2} + de_0e_4, e_0e_1, e_0e_2, e_0e_3\right) \qquad (18)\\ D' &= (-e_0e_0', -\langle e', e\rangle + d(e_0'e_4 - e_0e_4'), e_0'e_1 + e_0e_1', e_0'e_2 + e_0e_2', e_0'e_3 + e_0e_3') \end{aligned}$$

with $D, D' \in \mathbb{R}^5$. Now $\mathcal{H}_d(\hat{\mathcal{E}}) = \mathcal{S}_{D,D'}$ and one makes an analogous definition:

Definition 20. We define the hypersurface $\mathcal{V}_d(\hat{\mathcal{E}})$ as

$$\mathcal{V}_d(\hat{\mathcal{E}}) = \mathcal{S}_{\langle D,D'\rangle} \subset \mathbb{P}^4 \qquad (19)$$

where $\langle D, D'\rangle$ is the module quasi-generated by D and D'.

In this case also, $\mathcal{V}_d(\hat{\mathcal{E}}) \subset \mathcal{H}_d(\hat{\mathcal{E}})$ does not contain extraneous factors due to the parameters t where the rank of $W_{D,D'}(t)$ drops. At this point, it should be remarked that while we clearly always have

$$\mathcal{V}_d(\hat{\mathcal{E}}) \subset \mathcal{V}(\hat{\mathcal{E}}) \cap \{y_4 = -dy_0\}$$

this inclusion is not necessarily an equality (note that we had $\mathrm{Env}_d(\overline{\mathcal{E}}) = \mathcal{G}(\overline{\mathcal{E}}) \cap \{y_4 = -dy_0\}$ for the corresponding varieties). Analogously to Proposition 19 the following holds.

Proposition 21. *Let D_1, D_2 be a μ-basis of the module quasi-generated by D and D' and let k be the degree of the parametrization $D \wedge D'$ as in Section 2. Then*

$$\deg \mathcal{V}_d(\hat{\mathcal{E}}) = \deg D_1 + \deg D_2 = \deg(D \wedge D') - \deg q_{D,D'},$$

where $q_{D,D'} = \gcd(D \wedge D')$ and

$$\mathrm{Res}_t(D_1 \cdot (u, y_0, y_1, y_2, y_3)^T, D_2 \cdot (u, y_0, y_1, y_2, y_3)^T) = F^k_{\mathcal{V}_d(\hat{\mathcal{E}})}$$

where $F_{\mathcal{V}_d(\hat{\mathcal{E}})}$ is the implicit equation of $\mathcal{V}_d(\hat{\mathcal{E}})$.

Proof. It follows directly from Propositions 4 and 6. $\square$

Finally, we can use the correspondance of Proposition 17 to define the canal surface.

Definition 22. The Γ-hypersurface is defined as

$$\Gamma(\overline{\mathcal{E}}) = \Phi(\mathcal{V}(\hat{\mathcal{E}}) \cap \mathcal{Q}),$$

and the offset $\mathrm{Off}_d(\overline{\mathcal{E}})$ at distance d to the canal surface $\mathcal{C}$ is

$$\mathrm{Off}_d(\overline{\mathcal{E}}) = \Phi_0(\mathcal{V}_d(\hat{\mathcal{E}}) \cap \mathcal{Q}_d),$$

where $\mathcal{Q}_d = \{(u : y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^4 \mid -uy_0 + y_1^2 + y_2^2 + y_3^2 - d^2y_0^2 = 0\}$ and $\Phi_0(u, y_0, y_1, y_2, y_3) = (y_0, y_1, y_2, y_3)$. The canal surface itself is the special case $d = 0$ or in other words $\mathcal{C} = \mathrm{Off}_0(\overline{\mathcal{E}})$.

Note that the extraneous factors of $\mathcal{H}_d(\hat{\mathcal{E}})$ and $\mathcal{H}(\hat{\mathcal{E}})$ are in one-to-one correspondence with the extraneous factors of the corresponding hypersurfaces $\mathrm{Env}_d(\overline{\mathcal{E}})$ and $\mathcal{G}(\overline{\mathcal{E}})$ since they are caused by parameter values where the intersection of h_1 and h_2 (resp. g_1 and g_2) is of codimension one. So $\Gamma(\overline{\mathcal{E}})$ and $\mathcal{C}_d(\overline{\mathcal{E}})$ contain no such factors.

In this section and the previous one, many different geometric objects have been defined. We illustrate in the following diagram how they are related in order to make the situation clearer.

$$\begin{array}{ccccccc}
\mathbb{P}^4 & & \mathbb{P}^5 & & \mathbb{P}^5 & & \mathbb{P}^4 \\
\cup & & \cup & & \cup & & \cup \\
\mathcal{G}(\overline{\mathcal{E}}) & \xleftarrow{\Phi} & \mathcal{H}(\hat{\mathcal{E}}) \cap \mathcal{Q} & \supseteq & \mathcal{V}(\hat{\mathcal{E}}) \cap \mathcal{Q} & \xrightarrow{\Phi} & \Gamma(\overline{\mathcal{E}}) \\
\cup & & \cup & & \cup & & \cup \\
\mathrm{Env}_d(\overline{\mathcal{E}}) & \xleftarrow{\Phi_d} & \mathcal{H}_d(\hat{\mathcal{E}}) \cap \mathcal{Q}_d & \supseteq & \mathcal{V}_d(\hat{\mathcal{E}}) \cap \mathcal{Q}_d & \xrightarrow{\Phi_d} & \mathrm{Off}_d(\overline{\mathcal{E}}) \\
\cap & & \cap & & \cap & & \cap \\
\mathbb{P}^3 & & \mathbb{P}^4 & & \mathbb{P}^4 & & \mathbb{P}^3
\end{array} \tag{20}$$

Note that the hypersurfaces in the third row are included in the corresponding hypersurfaces in the second row. The first column is the naive definition of the objects to be studied: $\mathrm{Env}_d(\overline{\mathcal{E}})$ is more or less a d-offset to the canal offsets and $\mathcal{G}(\overline{\mathcal{E}})$ a hypersurface in one dimension higher containing all those offsets. However, they contain extraneous factors. So by passing to the second column, we linearize the hypersurfaces (i.e. we express them as resultants of linear forms) and can apply μ-bases to eliminate the extraneous factor, which gives the third column and finally go back down in dimension (by intersecting with $\mathcal{Q}$ and applying Φ to obtain the objects we are interested in: the offsets $\mathrm{Off}_d(\overline{\mathcal{E}})$ (in particular the canal surface $\mathcal{C} = \mathrm{Off}_0(\overline{\mathcal{E}})$) and the Γ-hypersurface.

5.1 The implicit equation.

We can now describe how to compute powers of the implicit equations of the dual varieties $\mathcal{V}(\hat{\mathcal{E}})$ and $\mathcal{V}_d(\hat{\mathcal{E}})$, the hypersurface $\Gamma(\overline{\mathcal{E}})$ and the offsets surface $\mathcal{C}_d(\overline{\mathcal{E}})$. We should remark that these powers (which are the degrees of the parametrizations of the corresponding Plücker curves) are in a way inherent to the geometry of the problem, as we shall illustrate in Example 27. They can be interpreted as the number of times the surface is traced by the spine curve. Note also that this not necessarily due to the non-properness of the spine curve: Even for a proper spine curve it can happen that the canal surface (or its offsets) is multiply traced, as in Example 27.

Algorithm (implicit equations)

1. INPUT: A rational vector $e(t) \in \mathbb{R}(t)^4$ as in formula (4).
2. Define $E, E' \in \mathbb{R}[t]^6$ as in formula (15) and $D, D' \in \mathbb{R}[t]^5$ as in formula (18).
3. Compute a μ-basis E_1, E_2 of the module $\langle E, E' \rangle$ and a μ-basis D_1, D_2 of the module $\langle D, D' \rangle$ using the algorithm in Section 2.

4. Set $F_{\mathcal{V}(\hat{\mathcal{E}})} = \mathrm{Res}\,_t(E_1 \cdot \hat{y}^T, E_2 \cdot \hat{y}^T)$ and $F_{\mathcal{V}_d(\hat{\mathcal{E}})} = \mathrm{Res}\,_t(D_1 \cdot (u, y_0, y_1, y_2, y_3)^T,$ $D_2 \cdot (u, y_0, y_1, y_2, y_3)^T) = 0$.

5. Let $F_{\Gamma(\overline{\mathcal{E}})}(y_0, y_1, y_2, y_3, y_4) = y_0^k F_{\mathcal{V}(\hat{\mathcal{E}})}((y_1^2+y_2^2+y_3^2-y_4^2)/y_0, y_0, y_1, y_2, y_3, y_4)$, where k is a minimal integer such that $F_{\Gamma(\overline{\mathcal{E}})}$ is a polynomial. Similarly, set
$F_{\mathcal{C}_d(\overline{\mathcal{E}})}(y_0, y_1, y_2, y_3) = y_0^k F_{\mathcal{V}_d(\hat{\mathcal{E}})}((y_1^2 + y_2^2 + y_3^2 - d^2 y_0^2)/y_0, y_0, y_1, y_2, y_3)$.

6. OUTPUT: $F_{\mathcal{V}(\hat{\mathcal{E}})}$, $F_{\mathcal{V}_d(\hat{\mathcal{E}})}$, $F_{\Gamma(\overline{\mathcal{E}})}$, and $F_{\mathcal{C}_d(\overline{\mathcal{E}})}$, which are powers of the implicit equation of the varieties $\mathcal{V}(\hat{\mathcal{E}})$, $\mathcal{V}_d(\hat{\mathcal{E}})$, $\Gamma(\overline{\mathcal{E}})$ and $\mathcal{C}_d(\overline{\mathcal{E}})$

Note that the affine parts of these equations can be obtained by replacing $y_0 = 1$ before the resultant computation.

5.2 The parametrization of the dual variety.

We can describe the parametrization of $\mathcal{V}(\hat{\mathcal{E}})$. The hyperplane defined by the equation

$$\det(\hat{y}, \hat{\mathcal{E}}(t)C, \hat{\mathcal{E}}'(t)C, a_1, a_2, a_3) = A_1 u + A_2 y_0 + A_3 y_1 + ... + A_6 y_4 = 0 \quad (21)$$

is tangent to the curve $\hat{\mathcal{E}}(t)C$, $a_i \in \mathbb{R}^6, i = 1, 2, 3$ are three arbitrary points. By the definition a point on the dual variety $\mathcal{V}(\hat{\mathcal{E}})$ is $(A_1, ..., A_6)$. Define $D = (\hat{\mathcal{E}}(t)C, \hat{\mathcal{E}}'(t)C, a_1, a_2, a_3)$ to be the 5×6 matrix with five rows $\hat{\mathcal{E}}(t)C, \hat{\mathcal{E}}'(t)C, a_1, a_2,$ a_3. And let $D_i, i = 1, ..., 6$ be 5×5 matrices obtained from D by removing the i-th column. Then using the Laplacian expansion by minors for the first row of the determinant (21) we obtain the parametrization of $\mathcal{V}(\hat{\mathcal{E}})$ as follows:

$$c(D) = (\det D_1, -\det D_2, \det D_3, -\det D_4, \det D_5, -\det D_6)/m \subset \mathcal{V}(\hat{\mathcal{E}}), \quad (22)$$

where $m = \gcd(D_1, ..., D_6)$. Here t, a_1, a_2, a_3 are arbitrary parameters.

6 The implicit degree of the hypersurface $\Gamma(\overline{\mathcal{E}})$.

The aim of this section is to get some formula for the implicit degree of the hypersurface $\Gamma(\overline{\mathcal{E}})$ in terms of the rational spine curve $\overline{\mathcal{E}} = \{\overline{e}(t) \in \mathbb{P}^4\}$. Notice that the implicit degree of the canal surface $\mathcal{C}$ is less or equal than $\deg \Gamma(\overline{\mathcal{E}})$ because we always have the inclusion

$$\mathrm{Off}\,_d(\overline{\mathcal{E}}) \subset \Gamma(\overline{\mathcal{E}}) \cap \{y_4 = -dy_0\}, \text{ i.e. } \deg \mathrm{Off}\,_d(\overline{\mathcal{E}}) \leq \deg \Gamma(\overline{\mathcal{E}}).$$

So this formula gives upper bound for the degree of the canal surface. In the case of a polynomial spine curve the upper bound was obtained in the paper [Xu et al.(2006)]. Note that for the computation of the implicit degree we do not need the implicit equation of the hypersurface. We believe that this formula is useful for higher degree spine curves because the computation of the implicit equation may be very difficult in practice.

Let us remind that the pre-image $\Phi^{-1}(\Gamma(\overline{\mathcal{E}})) \subset \mathcal{Q}$ is defined by the intersection of two varieties $\mathcal{V}(\hat{\mathcal{E}}) \cap \mathcal{Q}$. Let denote by $G(u, y_0, y_1, y_2, y_3, y_4) = 0$ the equation of $\mathcal{V}(\hat{\mathcal{E}})$. The Lie quadric has the equation $uy_0 = \langle y, y\rangle$ (recall that

$\langle y,y\rangle = y_1^2+y_2^2+y_3^2-y_4^2$). By the definition (22) the equation of $\Gamma(\mathcal{E})$ is obtained after the elimination of the variable u from the equations of $\mathcal{Q}$ and $\mathcal{V}(\hat{\mathcal{E}})$, i.e.

$$\Gamma(\overline{\mathcal{E}}) : \left\{ F(y_0,y_1,y_2,y_3,y_4) = y_0^k G\left(\frac{\langle y,y\rangle}{y_0}, y_0,y_1,y_2,y_3,y_4\right) = 0 \right\}, \tag{23}$$

where k is a minimal integer such that the left side of the equation (23) is polynomial. We introduce the following weighted degree

$$\begin{aligned} d_w(u^{k_1}y_0^{k_2}y_1^{k_3}y_2^{k_4}y_3^{k_5}y_4^{k_6}) &= 2k_1+k_3+k_4+k_5+k_6, \\ d_w(G(u,y_0,y_1,y_2,y_3,y_4)) &= \max_i\{d_w(m_i)\} \end{aligned} \tag{24}$$

where $G = \sum c_i m_i$ is a linear combination of the monomials m_i. Using this notation we have $\deg \Gamma(\overline{\mathcal{E}}) = d_w(G)$.

Let us assume that the curve $\mathcal{E} = \{e(t,s) \in \mathbb{R}^4 \mid (t,s) \in \mathbb{P}^1\}$ has a homogeneous parametrization. We introduce the following notations

$$w = (w_1,w_2,w_3,w_4),\ w_j = e_j' e_0 - e_j e_0',\ j=1,2,3,4, \tag{25}$$

$$\gamma = \max_j\{\deg(w_j,t)\}, \tag{26}$$

where e_i' means derivative with respect to t. We will say that the curve $\overline{\mathcal{E}} \in \mathbb{P}^4$ is of *general* type if :

$$\begin{aligned} &\gcd(w_1,w_2,w_3,w_4) = \gcd(e_0,e_0') = \gcd(e_0,\langle e,e\rangle) = 1,\ \deg\overline{\mathcal{E}} = \deg(e_0,t) \text{ and} \\ &\text{the parametrization degree of the Plücker curve } \varphi_{\mathcal{P}} : t \to E\wedge E' \text{ is one,} \end{aligned} \tag{27}$$

(i.e. $\deg \varphi_{\mathcal{P}} = 1$), where $e = (e_1,e_2,e_3,e_4)$ and E as in formula (15).

The first equation in the system (14) has the following form:

$$h_1 = \hat{\mathcal{E}} C \hat{y}^T = E\hat{y}^T = -e_0^2 u/2 - \langle e,e\rangle y_0/2 + e_0\langle e,y\rangle. \tag{28}$$

The polynomial h_1 is linear in the variables u,y_0,y_1,y_2,y_3,y_4 and has degree $2n$ in the variable t. The elimination of the variable t from the system $h_1 = h_1' = 0$ is the reducible polynomial $\mathrm{Res}_t(h_1,h_1') = H_1...H_kG$. By definition one of those factors is the equation of the dual variety $\mathcal{V}(\hat{\mathcal{E}}) = \{G\}$.

Proposition 23. *If $\overline{\mathcal{E}}$ is a curve of general type then $\deg \mathcal{V}(\hat{\mathcal{E}}) = 4n-2$, where $n = \deg\overline{\mathcal{E}}$. Moreover, we have $G\cdot LC(h_1) = \mathrm{Res}_t(h_1,h_1')$, where $LC(h_1)$ is the leading coefficient of the polynomial h_1 with respect to the variable t and $\{\hat{y} \in \mathbb{P}^5 \mid G(\hat{y}) = 0\} = \mathcal{V}(\hat{\mathcal{E}})$.*

Proof. For the curve of general type by the Proposition 19 we have $\deg \mathcal{V}(\hat{\mathcal{E}}) = \deg E\wedge E' - \deg(\gcd(E\wedge E'))$. We can compute components of the Plücker vector $E\wedge E' = ([1,2] : [1,3] : ... : [5,6]) \in \mathbb{P}^{14}$. For example, $[1,2+j] = e_0^2 w_j/2,\ j=1,2,3,4$ and $[2,3] = (-\langle e,e\rangle(e_0e_1)' + e_0e_1\langle e,e\rangle')/2$. By assumption (27) we see $\gcd([1,3],[1,4],[1,5],[1,6]) = e_0^2$ and $\gcd(e_0,[2,3]) = 1$. Therefore $\gcd(E\wedge E') = 1$. So we have $\deg\mathcal{V}(\hat{\mathcal{E}}) = \deg E\wedge E' = 4n-2$. Notice that $\deg \mathrm{Res}_t(h_1,h_1') = 4n-1$. Therefore we see $\deg G = \deg \mathrm{Res}_t(h_1,h_1')/LC(h_1) = 4n-2$, i.e. $G=0$ is the implicit equation of $\mathcal{V}(\hat{\mathcal{E}})$. □

Thereinafter, we will show that for the curve of general type, we have $d_w(G) = 6n-4$, where $n = \deg \overline{\mathcal{E}}$. For this we consider another resultant $\mathrm{Res}_t(h_1, h_2)$ and show that $d_w(\mathrm{Res}_t(h_1, h_2)) = d_w(G)$. We define h_2 in the following way. Let $g = e_0^2$ then we have the following equality $h_1' g - h_1 g' = e_0 h_2$, where

$$h_2 = (2\langle e, e\rangle e_0' - \langle e, e\rangle' e_0) y_0/2 + e_0 \langle e' e_0 - e e_0', y\rangle = (2\langle e, e\rangle e_0' - \langle e, e\rangle' e_0) y_0/2 + e_0 \langle w, y\rangle.$$

In other words, h_2 is the numerator of a rational function $\left(\frac{h_1}{g}\right)'$.

We often use the following properties of the resultant

$$\begin{aligned} \mathrm{Res}(f_1 f_2, h) &= \mathrm{Res}(f_1, h)\,\mathrm{Res}(f_2, h) \text{ (see [Cox et al.(1998)], p. 73)}, && (29)\\ \mathrm{Res}(f, h) &= a_0^{m-\deg r}\,\mathrm{Res}(f, r) \text{ (see [Cox et al.(1998)], p. 70)}, && (30) \end{aligned}$$

where $h = qf + r,\ \deg r \le \deg h = m, f = a_0 x^l + a_1 x^{l-1} + ... + a_l$.

We need an explicit formula for factors of the resultant

$$LC(h_1)\,\mathrm{Res}_{\,t}(h_1, h_1' g - h_1 g') = \mathrm{Res}_{\,t}(h_1, h_1' g), \tag{31}$$

where $LC(h_1)$ is a leading coefficient with respect to variable t of the polynomial h_1. Indeed, since $\deg(h_1, t) = \deg(g, t)$ then $\deg(h_1' g - h_1 g', t) + 1 = \deg(h_1' g, t)$. Thus we obtain the formula in (31) from the property (30). The left side of the formula (31) is equal to $LC(h_1)\,\mathrm{Res}_t(h_1, e_0 h_2) = LC(h_1)\,\mathrm{Res}(h_1, e_0)\,\mathrm{Res}_t(h_1, h_2)$. Otherwise, the right side of this formula is equal to $\mathrm{Res}_t(h_1, h_1')\,\mathrm{Res}_t(h_1, e_0)^2 = G \cdot LC(h_1)\,\mathrm{Res}_t(h_1, e_0)^2$.

Also, from the property (30) follows that $\mathrm{Res}_t(h_1, e_0) = y_0^{\deg(e_0, t)}$. Therefore we have and $d_w(\mathrm{Res}_t(h_1, h_2)) = d_w(G)$. In the lemma below, we prove that $d_w(\mathrm{Res}_t(h_1, h_2)) = 6n-4$.

We summarize our computations in the following

Theorem 24. *The degree of the hypersurface $\Gamma(\overline{\mathcal{E}})$ with the spine curve $\overline{\mathcal{E}}$ which satisfies the assumption (27) is equal to $6n-4$, where $n = \deg \overline{\mathcal{E}}$.*

Lemma 25. *With the notation as above we suppose that the conditions (27) are satisfied. Then we have the equality $d_w(\mathrm{Res}_{\,t}(h_1, h_2)) = 6n-4$, where $n = \deg \overline{\mathcal{E}}$.*

Proof. The weighted degree $d_w(G)$ may be viewed as a degree of a variety $\Phi(\{G\} \cap \mathcal{Q})$, where $\Phi : \mathbb{P}^5 \setminus q \to \mathbb{P}^4$ is a linear projection from the improper point $q = (1, 0, ..., 0)$ on the Lie quadric (see explicit formula (3)). The degree of the variety $\Phi(\{G\} \cap \mathcal{Q})$ can be computed constructively by counting points of intersection with a general line $L \in \mathbb{P}^4$. The pre-image of the line $C := \Phi^{-1}(L)$ is a conic on the Lie quadric $\mathcal{Q}$ which passes through the improper point q. Hence the degree of $\Phi(\{G\} \cap \mathcal{Q})$ is $2 \deg G - i(q, \{G\} \cap C)$, where $i(q, \{G\} \cap C)$ is the multiplicity of the intersection $\{G\} \cap C$ at the point q.

We need a parametric representation of the general conic $q \in C \subset \mathcal{Q}$. Assume that the conic C is in the parameterized plane $P : q + k_1 u + k_2 v$, where $k_1, k_2 \in \mathbb{R}^6$. The plane P intersects a singular cone $\{\langle x, x\rangle = x_1^2 + x_2^2 + x_3^2 - x_4^2\}$ on two lines. We choose two vectors k_1, k_2 in these lines so that the first coordinate is zero, i.e. $k_1 = (0, 1, a), k_2 = (0, 1, b), a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4)$

such that $\langle a,a\rangle = \langle b,b\rangle = 0$. With the notations as above the general conic $C := P \cap Q$ has the following parametrization:

$$C(u) := (ku-1, ku^2, ku^2 a + u(b-a)), \text{ where } k = 2\langle a,b\rangle. \tag{32}$$

Since $C(0) = q$ we can compute the multiplicity $m = i(q, \{\mathrm{Res}(h_1,h_2,t)\} \cap C)$ as follows. Lets denote by $ch_1 = h_1|_C, ch_2 = h_2|_C$ the restriction of polynomials h_1, h_2 to the conic C:

$$\begin{aligned} ch_1 &= -e_0^2(ku-1)/2 - kfu^2/2 + e_0\langle e, ku^2 a + u(b-a)\rangle, & (33)\\ ch_2 &= u((2fe_0' - f'e_0)ku/2 + e_0\langle w, kua + b - a\rangle), \text{ where } f = \langle e,e\rangle. & (34)\end{aligned}$$

The computation of the resultant gives $\mathrm{Res}_t(ch_1, ch_2) = u^m(A + Bu + ...)$, here $m = i(q, \{\mathrm{Res}_t(h_1,h_2)\} \cap C)$. On the other side we can compute the number of common points $(0, t_0)$ on curves $\{ch_1(u,t)\}$ and $\{ch_2(u,t)\}$ counted with multiplicities. This number coincides with m (see [Buse et al.(2005)], Proposition 5). The second curve $ch_2(u,t) = u \cdot ch_3(u,t)$ is reducible. Therefore, the resultant with respect to t is

$$\begin{aligned} \mathrm{Res}(ch_1, ch_2) &= \mathrm{Res}(ch_1, u)\,\mathrm{Res}(ch_1, ch_3) = u^{\deg(h_1,t)}\,\mathrm{Res}(ch_1, ch_3)\\ &= u^{2n}\,\mathrm{Res}(ch_1, ch_3)\end{aligned}$$

The second factor has a representation $ch_3 = uD(t) - N(t)$, where $D(t) = (2fe_0' - f'e_0)k/2 + e_0\langle w, ka\rangle$ and $N(t) = e_0\langle e'e_0 - ee_0', a - b\rangle$. It is easy to see that $\gcd(N(t), e_0^2) = e_0$. If $ch_1(0,t_0) = ch_3(0,t_0) = 0$ then $e_0(t_0) = 0$. The first curve $\{ch_1(u,t)\}$ is hyper-elliptic, i.e. the projection to t axes $pr : \{ch_1\} \to t$ is a map two-to-one. The curve $\{ch_1(u,t)\}$ has the following discriminant with respect to u:

$$\mathrm{disc}(ch_1, u) = e_0^2(\ (ke_0/2 - \langle e, b-a\rangle)^2 - 2e_0\langle e, ka\rangle + kf\ ). \tag{35}$$

It is ease to see that point $(0,t_0)$ is a singular point on the curve $\{ch_1(u,t)\}$ if and only if $e_0(t_0) = 0$. Therefore the point $(0,t_0)$ has multiplicity at least two as a point of the intersection of two curves $\{ch_1\} \cap \{ch_3\}$.

We will prove that the multiplicity of the intersection of two curves $\{ch_1\} \cap \{ch_3\}$ at the point $(0,t_0)$ equals to two if $e_0(t_0) = 0$. For simplicity we assume that $t_0 = 0$. The first equation (33) in the local ring $R = \mathbb{R}[u,t]_{\langle u,t\rangle}$ is

$$\begin{aligned} &\overline{ch}_1 = a_{21}u^2 t + a_{20}u^2 + a_{12}ut^2 - a_{11}ut + a_{02}t^2, \text{ where}\\ &[a_{21}, a_{20}, a_{12}, a_{11}, a_{02}] = [\tilde{e}_0 k\langle e,a\rangle, -fk/2, -\tilde{e}_0^2 k/2, \tilde{e}_0\langle e, a-b\rangle, 1/2\,\tilde{e}_0^2] \end{aligned} \tag{36}$$

and $t\tilde{e}_0 = e_0$. The second equation (34) in the local ring R is

$$\begin{aligned} &\overline{ch}_2 = u(\overline{ch}_3), \quad \overline{ch}_3 = b_{12}ut^2 + b_{11}ut + b_{10}u + b_{02}t^2 + b_{01}t,\\ &\text{where} \quad [b_{12}, b_{11}, b_{10}, b_{02}, b_{01}] =\\ &[\tilde{e}_0^2 k\langle e', a\rangle, -f'\tilde{e}_0 - \tilde{e}_0 e_0' k\langle e,a\rangle, kf\,(e_0' +\ e_0')\,, -\tilde{e}_0^2 k\langle e', a-b\rangle, \tilde{e}_0 e_0'\langle e, a-b\rangle].\end{aligned}$$

An easy computation with MAPLE shows that

$$\mathrm{Res}_t(\overline{ch}_1, \overline{ch}_3) = u^2(K_4u^4 + K_3u^3 + K_2u^2 + K_1u + K_0).$$

Therefore the point $(0,0)$ has multiplicity two if and only if $K_0 \neq 0$, i.e. K_0 is a unit in the local ring R. A straightforward computation shows that

$$K_0 = a_{02}\left({b_{10}}^2 a_{02} + {b_{01}}^2 a_{20} + b_{01}b_{10}a_{11}\right) = fk\tilde{e}_0^4 {e_0'}^2 \left(k\langle e, e\rangle + \frac{3}{4}\langle e, a-b\rangle^2\right).$$

Since $f(0) \neq 0$ and $e_0'(0) \neq 0$ by the condition (27) we conclude that $K_0 \neq 0$ for a general conic. Hence, the multiplicity $i(q, \{\mathrm{Res}_t(h_1, h_2)\} \cap \mathcal{Q})$ is equal to $\deg(h_1, t) + 2\deg(e_0, t) = 4n$. Therefore, we have

$$\begin{aligned} d_w(\mathrm{Res}_t(h_1, h_2)) &= 2\deg(\mathrm{Res}_t(h_1, h_2)) - i(q, \{\mathrm{Res}_t(h_1, h_2)\} \cap \mathcal{Q}) \\ &= 2(5n-2) - 4n = 6n - 4. \end{aligned}$$

□

Remark 26. We conjecture that the degree of the hypersurface $\Gamma(\overline{\mathcal{E}})$ is $\deg \mathcal{V}(\hat{\mathcal{E}}) + \deg(w) - \deg(\gcd(w))$, where w is defined by the formula (26).

7 Examples and special cases.

Let n be the degree of the spine curve $\mathcal{E} = \{e(t) \in \mathbb{R}^4\}$. And let $c_n(\mathcal{E})$ be the degree of the hypersurface $\Gamma(\overline{\mathcal{E}})$ with the spine curve $\mathcal{E}$.

Polynomial case. Assume that the spine curve is polynomial, i.e. $e_0 = 1$. By the theorem in [Xu et al.(2006)] the degree of hypersurface $\Gamma(\overline{\mathcal{E}})$ with the polynomial spine is at most $4n-2$.

For the general spine curve we have $c_n(\mathcal{E}) = 6n-4$, i.e. $c_n(\mathcal{E}) \leq 6n-4$. The lower bound is not clear. There are examples of spine curves with the following degrees:

$$\begin{aligned} c_2(\mathcal{E}) &= 3, 4, 5, 6, 8; \\ c_3(\mathcal{E}) &= 6, 7, 8, 9, 10, 11, 12, 14; \\ c_4(\mathcal{E}) &= 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20. \end{aligned}$$

It seems that there does not exist a spine curve such that $c_n(\mathcal{E}) = 6n-5$.

We consider three examples.

Example 27. Let us consider the following spine curve: $e(t) = \left(0, 0, \frac{8t}{1+t^2}, \frac{3-3t^2}{1+t^2}\right)$, $n = 2$. This is a proper parametrization of an ellipse in $\mathbb{R}^4$. We find the Plücker coordinate vector $P = \hat{\mathcal{E}}(t)C \wedge \hat{\mathcal{E}}'(t)C$ and $q = \gcd(P) = 1$. Also, we see that $\deg \mathcal{V}(\hat{\mathcal{E}}) = \deg P - \deg(q, t) = 8$ and $\gamma = \deg(w) = 2$. If we run the μ-basis algorithm with two input vectors $\mathcal{E}(t)C, \mathcal{E}'(t)C$ we get the output two vectors E_1 and E_2:

$$\begin{aligned} E_1 \cdot \hat{y}^T &= 4\,t^3 y_3 + (-u - 41\,y_0)\,t^2 + 12\,t y_3 + 9\,y_0 - u - 6\,y_4, \\ E_2 \cdot \hat{y}^T &= (u - 9\,y_0 - 6\,y_4)\,t^3 - 12\,t^2 y_3 + (41\,y_0 + u)\,t - 4\,y_3. \end{aligned}$$

Now we can find the implicit equation $G = \mathrm{Res}(E_1 \cdot \hat{y}^T, E_2 \cdot \hat{y}^T, t)$ of the dual variety $\mathcal{V}(\hat{\mathcal{E}})$. The polynomial G contains 26 monomials and has degree 6 (as in Proposition 23). The equation of the hypersurface $\Gamma(\overline{\mathcal{E}})$ is defined by the

polynomial
$F(y_0,...,y_4) = y_0^2 G(\langle y,y\rangle/y_0, y_0, y_1, y_2, y_3, y_4)$ of degree 8. Since,

$$F(1,y_1,y_2,y_3,0) = \left({y_1}^2+16+{y_2}^2+8\,y_3+{y_3}^2\right)\left({y_1}^2+{y_2}^2+16-8\,y_3+{y_3}^2\right)\left(-225+25\,{y_1}^2+25\,{y_2}^2+9\,{y_3}^2\right)^2,$$

the 0-envelope of the canal surface $\mathrm{Env}_0(\mathcal{E}) = \Gamma(\overline{\mathcal{E}}) \cap \{y_4 = 0\}$ is reducible. The canal surface $\mathcal{C}$ is the double ellipsoid of revolution $(-225+25\,{y_1}^2+25\,{y_2}^2+9\,{y_3}^2)^2$. Indeed, for the computation of $\mathcal{C}(\overline{\mathcal{E}})$ we should assume that the variable $y_4 = 0$ and to repeat the same steps as above. We should consider only the first 5 coordinates of the vectors $\mathcal{E}(t)C, \mathcal{E}'(t)C$. Let us denote these two vectors with 5 coordinates by D_1, D_2. But this time we see that the Plücker vector $\hat{P} = D_1 \wedge D_2$ has a non-trivial common divisor, i.e. $\hat{q} = \gcd(\hat{P}) = t^2-1$. So, using the μ-basis algorithm we find the μ-basis R_1, R_2 for the input D_1, D_2. In this case we see that $\deg R_1 = \deg R_2 = 2$. Now we find the resultant $\check{G} = \mathrm{Res}_t(R_1\cdot\check{y}^T, R_2\cdot\check{y}^T) = \left(16\,{y_3}^2+225\,{y_0}^2-25\,y_0u\right)^2$, where $\check{y} = (u, y_0, y_1, y_2, y_3)$. After the substitution $u = (y_1^2+y_2^2+y_3^2)/y_0$ we obtain the implicit equation of the canal surface the double ellipsoid $(-225+25\,{y_1}^2+25\,{y_2}^2+9\,{y_3}^2)^2$. We can see this geometrically, too. The point $e(t) \in \mathbb{R}^4$ corresponds to the sphere $S(e(t)) \in \mathbb{R}^3$ with a center on the y_3-axis. If $t \in [-1/2, 1/2]$ then the sphere $S(e(t))$ is tangent to the ellipsoid $EL = (-225+25\,{y_1}^2+25\,{y_2}^2+9\,{y_3}^2)$, and inside this ellipsoid. Moreover, the real envelope of the family $S(e(t)), t \in [-1/2, 1/2]$ is the ellipsoid EL. Note that the sphere $S(e(1/t))$ has the same center but the opposite radius to the sphere $S(e(t))$, i.e. it has the opposite orientation. Therefore, the real envelope of the family $S(e(t)), t \in (-\infty, -2] \cap [2, \infty)$ is the same ellipsoid EL. Hence, from the point of Laguerre geometry the envelope of the whole family $S(e(t))$ is the double ellipsoid EL^2. Note, that the d-offset to the canal surface, in this case is the d-offset to ellipsoid EL and it has degree 8. Also, we can check that by Theorem 24 the degree of the $\Gamma(\overline{\mathcal{E}})$ hypersurface is 8, too. For a detailed study and other examples of canal surfaces with a quadratic spine curve we recommend to look at the paper [Krasauskas, Zube(2007)].

Example 28. Consider the polynomial spine curve $e(t) = \left(3t^2+1, 4t^2+t, 0, 5t^2\right)$, $n = 2$. We find the Plücker coordinate vector $P = \hat{\mathcal{E}}(t)C \wedge \hat{\mathcal{E}}'(t)C$ and $q = \gcd(P) = 1$. Also, we see that $\deg \mathcal{V}(\hat{\mathcal{E}}) = \deg P - \deg q = 4$ and $\gamma = \deg(w) = 1$. If we run the μ-basis algorithm with two input vectors $\hat{\mathcal{E}}(t)C, \hat{\mathcal{E}}'(t)C$ we get the output of two vectors

$$\begin{aligned} E_1 &= -u + \left(-1-7\,t^2-8\,t^3\right)y_0 + \left(2+6\,t^2\right)y_1 + 2\,t\,(1+4\,t)\,y_2 - 10\,t^2 y_4, \\ E_2 &= -t\,(7+12\,t)\,y_0 + 6\,ty_1 + (1+8\,t)\,y_2 - 10\,ty_4, \end{aligned}$$

and find the implicit equation G of the dual variety $\mathcal{V}(\hat{\mathcal{E}})$ (it contains 54 monomials, so we do not present an explicit formula). Finally, we find that $c_2(\mathcal{E}) = d_w(G) = 5$. For this example, we have $\deg \mathcal{C} = \deg \Gamma(\overline{\mathcal{E}})$, i.e. the implicit degree of the canal surface is 5. Note that this contradicts Theorem 4 in [Xu et al.(2006)], because in this example the degree of the canal surface is an odd number. It seems that the mentioned theorem gives only an upper bound estimation, but not the exact degree formula of canal surfaces with polynomial spine curve.

Example 29. In the next example we take the following spine curve:
$e(t) = \left(\frac{(1-t^2)^2}{(1+t^2)^2}, 2\,\frac{t(1-t^2)}{(1+t^2)^2}, 2\,\frac{t}{1+t^2}, 1 \right)$, $n = 4$. The first three coordinates define the Viviani curve, i.e. it is intersection curve of the sphere and the tangent cylinder. We find the Plücker coordinate vector $P = \hat{\mathcal{E}}(t)C \wedge \hat{\mathcal{E}}'(t)C$ and $q = \gcd(P) = 1$. Also, we see that $\deg \mathcal{V}(\hat{\mathcal{E}}) = \deg P - \deg q = 6$. If we run the μ-basis algorithm with two input vectors $\hat{\mathcal{E}}(t)C, \hat{\mathcal{E}}'(t)C$ we get output of two vectors

$$\begin{aligned} E_1 &= \left(0, 4+4\,t^2, 4-4\,t^2, 6\,t-2\,t^3, 6\,t+2\,t^3, 4+4\,t^2\right), \\ E_2 &= \left(0, 4\,t+4\,t^3, 4\,t\left(-1+t^2\right), 2-6\,t^2, 2+6\,t^2, 4\,t+4\,t^3\right), \end{aligned}$$

both of degree 3 and find the implicit equation G of the dual variety $\mathcal{V}(\hat{\mathcal{E}})$ (it contains 58 monomials). Finally, we find that $c_4(\mathcal{E}) = d_w(G) = 10$. For this example, we have $\deg \mathcal{C} = \deg \Gamma(\overline{\mathcal{E}})$, i.e. the implicit degree of the canal surface is 10.

References

[Buse et al.(2005)] L.Buse, H. Khalil, B. Mourrain Resultant-based methods for plane curves intersection problems, Proceedings of the CASC'2005 conference, Lecture Notes in Computer Science, Vol. 3718 (2005), pp. 75-92.

[Cecil(1992)] T.E. Cecil, Lie Sphere Geometry, Springer, 1992.

[Chen(2003)] F.Chen, Reparametrization of a rational ruled surface using the μ-basis., Computer Aided Geometric Design, 20 (2003), 11–17.

[Chen, Wang(2003)] F.Chen, W.Wang, The mu-basis of a planar rational curve — properties and computaion, Graphical Models, 64 (2003), 368–281.

[Chen et al.(2001)] F.Chen, J.Zheng, T.W.Sederberg, The mu-basis of a rational ruled surface, Computer Aided Geometric Design, 18 (2001), 61–72.

[Cox, Sederberg, Chen (1998)] D.Cox, T.W. Sederberg, F. Chen, The moving line ideal basis for planar rational curves. Computer Aided Geometric Design, 15, (1998), 803-827.

[Cox et al.(1998)] D. Cox, J. Little, D. O'Shea, Using Algebraic Geometry, 1998, Springer.

[Degen(2002)] W.Degen, Cyclides, in Handbook of Computer Aided Geometric Design, 2002, p.575–601

[Dohm(2006)] M. Dohm, Implicitization of rational ruled surfaces with μ-bases, accepted for publication in Journal of Symbolic Computation, Special Issue EACA 2006, preprint available at http://arxiv.org/pdf/math/0702658

[Kazakeviciute(2005)] M.Kazakeviciute "Blending of natural quadrics with rational canal surfaces", PhD thesis, Vilnius University, 2005.

[Krasauskas(2007)] R. Krasauskas, Minimal rational parametrizations of canal surfaces, Computing, vol.79 (2007), 281-290.

[Krasauskas, Mäurer(2000)] R. Krasauskas, C. Mäurer, Studying cyclides using Laguerre geometry, *Computer Aided Geometric Design* **17** (2000) 101–126.

[Krasauskas, Zube(2007)] R. Krasauskas, S. Zube, Canal Surfaces Defined by Quadratic Families of Spheres, in: Geometric Modeling and Algebraic Geometry, B. Jttler, R. Piene (Eds.), Springer, 2007, pp. 138-150. (Publication: November, 2007)

[Landsmann et al.(2001)] G. Landsmann, J. Schicho and F. Winkler, The parametrization of canal surfaces and the decomposition of polynomials into a sum of two squares, *J. Symbolic Computation* **32** (2001) 119–132.

[Peternell, Pottmann(1997)] M. Peternell and H. Pottmann, Computing rational parametrizations of canal surfaces, *J. Symbolic Computation* **23** (1997) 255–266.

[Pottmann, Peternell(1998)] H. Pottmann and M. Peternell, Application of Laguerre geometry in CAGD, *Computer Aided Geometric Design* **15** (1998) 165–186.

[Pottmann et al.(1998)] H. Pottmann, M. Peternell, B. Ravani, Contributions to computational line geometry, *in Geometric Modeling and Procesing '98* (1998) 43–81.

[Pratt(1990)] M. J. Pratt, Cyclides in computer aided geometric design, *Computer Aided Geometric Design* **7** (1990) 221–242.

[Pratt(1995)] M. J. Pratt, Cyclides in computer aided geometric design II, *Computer Aided Geometric Design* **12** (1995) 131–152.

[Segundo, Sendra(2005)] F. San Segundo, J. R. Sendra, Degree Formulae for Offset Curves, Journal of Pure and Applied Algebra, vol 195/3, (2005) 301-335.

[Xu et al.(2006)] Z. Xu, R. Feng and J. Sun, Analytic and algebraic properties of canal surfaces, Journal of Computational and Applied Mathematics, 195(1-2), (2006) 220-228.